Stanford University Library
APR 21 1903

APRIL 1903

ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

EDITED BY

ORMOND STONE W. E. BYERLY H. S. WHITE
F. S. WOODS MAXIME BÔCHER
E. V. HUNTINGTON

PUBLISHED UNDER THE AUSPICES OF HARVARD UNIVERSITY

SECOND SERIES Vol. 4 No. 3

FOR SALE BY
THE PUBLICATION OFFICE OF HARVARD UNIVERSITY
2 University Hall, Cambridge, Mass., U. S. A.

London: LONGMANS, GREEN & CO.
39 Paternoster Row

Leipzig: OTTO HARRASSOWITZ
Querstrasse 14

THE CARDIOID AND TRICUSPID: QUARTICS WITH THREE CUSPS

By R. C. Archibald

1. The *cardioid* and the three-cusped hypocycloid or *tricuspid* are of interest not only on account of the great many beautiful properties which they possess, but also because of their historic association with the names of many eminent mathematicians.

The cardioid was discussed as an epicycloid by Jacob Ozanam as early as 1691,* and later by the Bernoullis, de L'Hôpital, L. Carré (author of the first complete work on the integral calculus, 1700), de Réaumur, de la Hire, Maclaurin, Castilleoneus, Euler, Cramer, Quetelet (who showed the uses of the curve in graphic astronomy), Magnus (who showed the curve to be an inverse of the parabola), J. C. Maxwell, R. Proctor, Wolstenholme, Weill, Laguerre, Brocard, etc.

The tricuspid seems to have been first conceived, as a unicursal quartic, by Euler,† in 1745, in the treatment of a problem in catacaustics. Some idea of the interest which this curve has excited in the minds of many of the most noted modern mathematicians, and of the enormous number of papers devoted to the discussion of it, may be obtained by glancing through the bibliographical lists in *L'Intermédiaire des Mathématiciens*, vol. 3 (1896) and vol. 4 (1897).

The object of the present paper is to show the intimate relation which exists between these apparently entirely dissimilar curves, and, in illustration of the theory of projection, to make this relation a basis for the derivation of some theorems for the general tricuspidal quartic. Incidentally, we shall see how a comparatively simple property of one curve may be made use of to prove a property of the other which could be otherwise derived only by much longer and less elegant methods.

The subject has been touched upon‡ by Siebeck, Frahm, Wolstenholme,

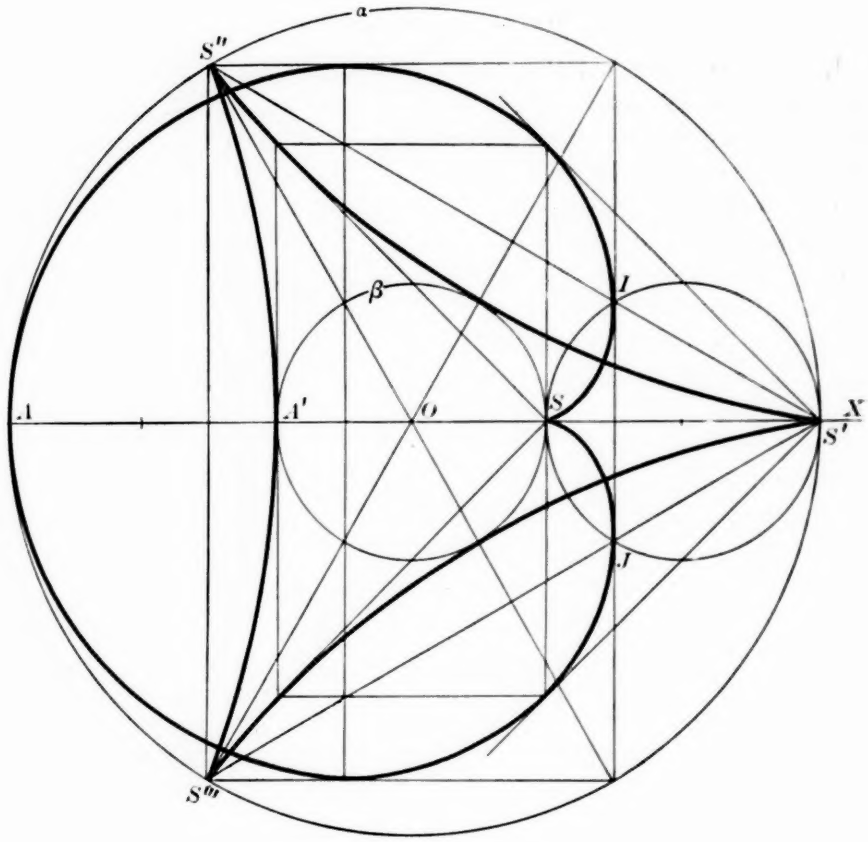
---

* J. Ozanam, *Dictionaire Mathématique ou Idée générale de Mathématique*, pp. 102–104 (Amsterdam, 1691).

† P. H. Fuss, *Correspondance math. et phys. du XVIII^{ème} siècle*, vol. 1 (Euler and Goldbach), pp. 314–488 (published in 1843). See also *Acta eruditorum*, 1746.

‡ Siebeck, *Crelle's Journal*, vol. 66 (1866), p. 359. — Frahm, *Schlömilch's Zeitschrift für Math.*, vol. 18 (1873), p. 363. — Wolstenholme, *Proc. London Math. Soc.*, vol. 4 (1873), p. 330;

Townsend and Delens; but it is hoped that the presentation of their results in a somewhat new and connected form, with the addition of several new theorems, may be suggestive and of interest.



Numerous properties of the cardioid and tricuspid will be assumed as known. For the former, reference may be made to my Dissertation; for the latter, to the bibliographies already cited.

Educational Times Reprint, vol. 55 (1891), p. 182; *Mathematical Problems*, 3rd edit. (1891), problem no. 1849, and note p. 251.—Townsend, *Educ. Times Reprint*, vol. 20 (1873), p. 34; vol. 22 (1875), p. 35.—Delens, *Journal de math. spéciales*, vol. 15 (1892), pp. 193–198.

* Archibald, *The Cardioide and some of its related curves*, Strassburg, 1900.

2. Consider a cardioid, C, and a tricuspid, T, placed in a very simple relation to each other, as follows:

Let $OA = 3a$ and $OA' = a$ be the radii of two concentric circles, α and β, respectively (see figure). Then:

If a circle with radius a roll on the outside of β, a point on its circumference will trace the *cardioid*, C, of the figure.*

The equations of C, referred to O as origin, are:

$$\begin{cases} x = a(2\cos\theta - \cos 2\theta), \\ y = a(2\sin\theta - \sin 2\theta). \end{cases}$$

Eliminating θ, we get

$$(x^2 + y^2)^2 - 6a^2(x^2 + y^2) + 8a^3x - 3a^4 = 0.$$

Introducing trilinear coordinates, this equation becomes:

If a circle with radius a roll on the inside of α, a point on its circumference will trace the *tricuspid*, T, of the figure.*

The equations of T, referred to O as origin, are:

$$\begin{cases} x = a(2\cos\theta + \cos 2\theta), \\ y = a(2\sin\theta - \sin 2\theta). \end{cases}$$

Eliminating θ, we get

$$(x^2 + y^2)^2 + 18a^2(x^2 + y^2) - 8ax(x^2 - 3y^2) - 27a^4 = 0.$$

Introducing trilinear coordinates, this equation becomes:

$$X^{-\frac{1}{2}} + Y^{-\frac{1}{2}} + Z^{-\frac{1}{2}} = 0, \qquad (A)$$

where

$$X = x + iy - a,$$
$$Y = x - iy - a,$$
$$Z = -a;$$

or,

$$\text{(I)} \quad \begin{aligned} x &= (X + Y - 2Z)/2, \\ y &= (X - Y)/2i, \\ a &= -Z. \end{aligned}$$

Equation (A) is the trilinear equation of C referred to an imaginary triangle, whose vertices are the three cusps of C; viz., the point $(a, 0)$ and the circular points at infinity.

where

$$X = x + \sqrt{3}y - 3a,$$
$$Y = x - \sqrt{3}y - 3a,$$
$$Z = -3a - 2x;$$

or,

$$\text{(II)} \quad \begin{aligned} x &= (X + Y - 2Z)/6, \\ y &= (X - Y)/2\sqrt{3}, \\ a &= -(X + Y + Z)/9. \end{aligned}$$

Equation (A) is the trilinear equation of T referred to a real triangle, whose vertices are the three cusps of T; viz., the real points $(3a, 0)$, $(-3a/2, 3\sqrt{3}a/2)$, and $(-3a/2, -3\sqrt{3}a/2)$.

---

* The following relations between C and T may be worth noting: The axis (SA) of C is equal to the axis $(S'A')$ of T, and the length, $16a$, of the curve C is equal to the length of the curve T. The cuspidal chords of C are of constant length, $4a$, and the locus of their middle points is the base, β, of C. The portions of tangents to T intercepted by the curve are of constant length, $4a$, and the locus of their middle points is also the circle β.

3. By the aid of relations (I) and (II), the trilinear equation of any curve, referred to either of the triangles just described, is readily obtained from its equation in rectangular coordinates. For example:

---

1. From the equation $x = 3a/2$ of the double tangent of C, we get

1′. From the equation $a = 0$ of the line at infinity, we get

$$X + Y + Z = 0. \qquad (B)$$

---

2. For the circle

$$(x - 2a)^2 + y^2 = 0$$

through the cusp and the points of contact, I, J, of the double tangent, we get

2′. For the circle **a**:

$$x^2 + y^2 = 9a^2,$$

through the three cusps, we get

$$XY + YZ + ZX = 0,$$

or,

$$(X + Y)^{\frac{1}{2}} + (Y + Z)^{\frac{1}{2}} + (Z + X)^{\frac{1}{2}} = 0. \qquad (C)$$

---

3. For the cuspidal tangents, $y = 0$, $x + iy = 0$, $x - iy = 0$, we get

3′. For the cuspidal tangents, $y = 0$, $y + \sqrt{3}x = 0$, $y - \sqrt{3}x = 0$, we get

$$X - Y = 0, \quad Y - Z = 0, \quad Z - X = 0.$$

These three lines intersect at the focus, O, of the cardioid.

These three lines intersect at the focus, O, of the tricuspid.

---

4. For the circle **a**:

$$x^2 + y^2 = 9a^2,$$

we get

4′. For the ellipse

$$9(x + 2a)^2 + y^2 = 9a^2,$$

we get

$$XY - YZ - ZX = 8Z^2,$$

or,

$$(X - Z)^{\frac{1}{2}} + (Y - Z)^{\frac{1}{2}} + (X + Y - 8Z)^{\frac{1}{2}} = 0, \qquad (D)$$

or,

$$(X - Z)(Y - Z) = 9Z^2.$$

5. For the hyperbola
$$25(x + 6a/5)^2 - 135y^2 = 81a^2,$$
we get

5′. For the circle β:
$$x^2 + y^2 = a^2,$$
we get

$$8(X + Y + Z)^2 = 27(XY + YZ + ZX),$$
or,
$$(X + Y - 8Z)^{\frac{1}{2}} + (Y + Z - 8X)^{\frac{1}{2}} + (Z + X - 8Y)^{\frac{1}{2}} = 0, \qquad (E)$$
or,
$$8(X^2 + Y^2 + Z^2) = 11(XY + YZ + ZX).$$

---

6. For the circle β:
$$x^2 + y^2 = a^2,$$
we get

6′. For the ellipse
$$25(x - 6a/5)^2 - 15y^2 = 81a^2,$$
we get

$$XY - YZ - ZX = 0. \qquad (F)$$

---

7. For the circle
$$(3x - 2a)^2 + 9y^2 = a^2,$$
we get

7′. For the hyperbola
$$49(x - 12a/7)^2 - 63y^2 = 81a^2,$$
we get

$$3XY - YZ - ZX = 0. \qquad (G)$$

---

4. It has been shown by Salmon* that the trilinear equation of any tricuspidal quartic referred to a suitable triangle of reference is

$$X^{-\frac{1}{2}} + Y^{-\frac{1}{2}} + Z^{-\frac{1}{2}} = 0.$$

Since either of two curves having the same trilinear equation but different triangles of reference may be obtained from the other by projection, it follows that from either the cardioid or the tricuspid we may, by a suitable projection, obtain the other;† and further that from either we may obtain any tricuspidal quartic.‡

---

* G. Salmon, *Higher Plane Curves*, 3rd edit. (1879), p. 258.

† The projective transformation which transforms C into T is given by the equations:

$$x = \frac{3ax'}{2x' + 3a}, \qquad y = \frac{-\sqrt{3}\,iay'}{2x' + 3a}.$$

‡ W. K. Clifford, in a paper: On Triangular Symmetry [*Educ. Times Reprint*, vol. 4 (1866), pp. 88-89; *Collected Math. Papers* (1882), pp. 412-414], attributes to Cayley a proof that "every three-cusped quartic can be projected into a hypocycloid of three cusps."

Notice that in any projection of the cardioid the circular points at infinity ($Y = Z = 0$, $X = Z = 0$) become cusps of the tricuspidal quartic into which the cardioid projects, while the line at infinity ($Z = 0$) becomes the line joining the cusps in question: on the other hand, in any projection of the tricuspid the line at infinity ($X + Y + Z = 0$) becomes the double tangent of the tricuspidal quartic into which the tricuspid projects, while the circular points at infinity become the points of contact I, J of the double tangent. Keeping these facts in mind, it is easy to obtain from any descriptive property of the cardioid [tricuspid]— or from any metric property which lends itself to projection—a property of any tricuspidal quartic, and incidentally of the tricuspid [cardioid].

5. Among many remarkable known properties of the cardioid and tricuspid we shall note a few which readily lend themselves to projection.

On the cardioid:

(a) Tangents to C at the ends of a cuspidal chord intersect orthogonally on the circle α.

(b) Normals to C at the ends of a cuspidal chord intersect orthogonally on the circle β.

(c) The locus of a point which so moves that the points of contact of tangents from it to C lie on a line, Δ, is the circle $(x - 2a)^2 + y^2 = a^2$,* passing through the cusp and the points of contact I, J of the double tangent.

(d) The locus of a point which so moves that the feet of the normals drawn from it to C lie in a line, Δ', is the circle $(3x - 2a)^2 + 9y^2 = a^2$.*

On the tricuspid:

(e) Tangents to T at the points where any tangent t cuts the curve, intersect orthogonally in a point P on the circle β; the third tangent which can be drawn from P to T meets t orthogonally on β.

(f) If three tangents are drawn to T from any point P on the circle β, the normals at their points of contact will intersect on the circle α at the point where α is cut by the line PO.

(g) Normals to T at the points where a tangent to T cuts the curve intersect orthogonally on the circle α.

(h) The locus of the point which so moves that the feet of normals drawn from it to T lie in a line, is the circle α.

6. By the aid of the relations established in §§ 2 – 4, the reader will

*R Lachlan, *Educ. Times Reprint*, vol. 58 (1893), p. 42.

find it easy to extend these theorems of §5 to any tricuspidal quartic. For instance the following theorems are easily deduced:

(1°) The locus of a point which so moves that the points of contact of tangents from it to any tricuspidal quartic $X^{-\frac{1}{2}} + Y^{-\frac{1}{2}} + Z^{-\frac{1}{2}} = 0$ lie in a line Δ, is the conic (C') : $XY + YZ + ZX = 0$, through the cusps of the quartic and the points of contact I, J of its double tangent. The lines joining the triple focus of the quartic to I and J are tangent to this conic. For the special case of the tricuspid the locus in question is the circle a through the cusps.

(2°) Tangents to any tricuspidal quartic $X^{-\frac{1}{2}} + Y^{-\frac{1}{2}} + Z^{-\frac{1}{2}} = 0$ at the points P_2, P_3 when any tangent t cuts the curve, intersect in a point P on the conic (E') : $8(X + Y + Z)^2 = 27\,(XY + YZ + ZX)$, which touches the quartic in the points where the quartic is cut by its cuspidal tangents, and passes through I and J, the points of contact of the double tangent. These tangents PP_2 and PP_3 divide the line IJ harmonically. The third tangent from P to the quartic meets t in a point Q on (E'); the two tangents PQ and t also divide IJ harmonically. The line PO cuts the tangent t in its second point of intersection with (E'). Further, OI and OJ are tangent to (E'); hence the focus of the quartic is the pole of the double tangent with respect to either of the conics (C') or (E').

For the special case of the cardioid the conic (E') is a hyperbola.*

(3°) Quasi-normals† with respect to a pair of cusps B, C of a tricuspidal quartic, drawn at the ends of a chord through the third cusp, meet on the conic (F') : $XY - YZ - ZX = 0$ (which passes through the cusps), and divide BC harmonically.

(4°) If the feet of the three quasi-normals (with respect to I, J) which can be drawn from a point to a tricuspidal quartic lie in a line Δ', the locus of the point is a conic (G') : $3XY - YZ - ZX = 0$, passing through the cusps. For the special case of the cardioid the locus in question is a circle.

7. It is not difficult to prove that the envelope of the line Δ through the points of contact of the tangents drawn to the tricuspid from any point P

---

* A result first given in 1866, by Siebeck, *loc. cit.*; but also in 1874 by Wolstenholme, *Educ. Times Reprint*, vol. 20, p. 34.

† A *quasi-normal* with respect to a pair of points is a line through the point of contact, P_1, of a tangent and forming with the tangent and the lines drawn from P_1 to the two points a harmonic pencil. A *quasi-evolute* is the envelope of the quasi-normals. Obviously a normal will always project into a quasi-normal with respect to the projections of the circular points at infinity, and an evolute into a quasi-evolute.

on the circle a (§ 5(c)) is the evolute of the tricuspid.* This curve may be obtained by enlarging T three times its linear dimensions and turning the resulting figure through 180°. The equation of this envelope is:

$$(X + 4Y + 4Z)^{-\frac{1}{2}} + (4X + Y + 4Z)^{-\frac{1}{2}} + (4X + 4Y + Z)^{-\frac{1}{2}} = 0. \quad (H)$$

Hence we have the theorem:

(5°) For any tricuspidal quartic, the envelope of the line Δ is another tricuspidal quartic (H'), tangent to the conic (C') at each of the cusps of the original quartic, and having the same cuspidal and double tangents as the original quartic, and the same points of contact I, J. Or, the envelope of Δ is the quasi-evolute (H') of the quartic, with respect to the points of contact I, J of the double tangent of the quartic.

We may finally deduce the following theorem from §5 (f), (g) with regard to the conic (C''):

(6°). The quasi-normals of a tricuspidal quartic (with respect to the points I, J) at three points P_1, P_2, P_3 intersect in a point Q on the conic (C'') which passes through the cusps of the quartic; the quasi-normals QP_2, QP_3 divide IJ harmonically. Further, the points P, O, Q lie in a straight line which passes through one of the points where the tangent P_2P_3 cuts the conic (E').

8. *Problems.*

1. The points P_2, P_3 in §6 (2°) are said to be conjugate. Show that in the case of a cardioid these points lie on a circle through the real cusp S and the focus O of the curve; further that the tangents PP_2, PP_3 intersect the double tangent on this circle.† The equation of the circle being $x^2 + y^2 - ax - by = 0$ (b variable), deduce a general theorem for tricuspidal quartics.

2. The *orthoptic locus* of C is made up of the circle a (§5(a)) and the Limaçon of Pascal:

$$8(x^2 - y^2 - 9a^2)^2 + 54a^2(x^2 + y^2 - 9a^2) + 81(2x - 3a)a^3 = 0.‡$$

Hence, show that, if through a cusp S of any tricuspidal quartic $X^{-\frac{1}{2}} + Y^{-\frac{1}{2}} + Z^{-\frac{1}{2}} = 0$ any line be drawn, it meets the curve in two other points P_2, P_3, the tangents at which divide the line BC (joining the other cusps) harmonically; the locus of the intersection of these tangents is the conic (D'). The

* Delens, *loc. cit.*

† *Cf.* Archibald, *Educ. Times Reprint*, problem no. 14586, vol. 75 (1901), p. 127.

‡ Wolstenholme, *Proc. London Math. Soc.*, *loc. cit.*

locus of the point of intersection of other tangents to the quartic which divide BC harmonically, is the bicuspidal quartic

$$8(X-Z)^2(Y-Z)^2-9Z^2(X-Z)(Y-Z)-81XYZ^2=0,$$

which is tangent to the tricuspidal quartic at the two points where the quasi-normals are also tangents.

3. With the notation of the last problem, show that the locus of the fourth harmonic point to P_2, S, P_3 is a cuspidal cubic with an inflexional tangent, defined by the equation $XY(X+Y)=Z(X-Y)^2$. Show that for the cardioid C, this cubic is the *Cissoid of Diocles*, with cusp at S and inflexional asymptote $x=3a$.*

4. The envelope of the sides of the triangles† formed by joining the points of contact of parallel tangents to a cardioid C, is the nodal cubic known as the *Trisectrix of Maclaurin*‡. Its equation may be written

$$r\sin 2\theta + a\sin 3\theta = 0, \quad \text{and} \quad r\cos(\theta/3) = a/2.$$

Hence, deduce a theorem for tricuspidal quartics.

9. Beside the cardioid and tricuspid, other definite examples of tricuspidal quartics have occurred in problems by Cayley§ and Wolstenholme.‖ To quote Problem 1698 of the latter may not be without interest in this connection :

"Through each point within a parabola $y^2=4ax$ it is obvious that at least one minimum chord can be drawn : prove that the part from which two minimum chords and one maximum can be drawn is divided from the part through which only one minimum can be drawn by the curve $(x-5a)^{-\frac{1}{3}}+(4x-6y+4a)^{-\frac{1}{3}}+(4x+6y+4a)^{-\frac{1}{3}}=0$; etc."

Euler found (*loc. cit.*) a whole class of tricuspidal quartics in connection with a solution of the problem : "given a luminous point, find a curve such that rays of light twice reflected from it return to the starting point."

---

* Archibald, *Educ. Times Reprint*, vol. 74 (1901), p. 53.

† These triangles have a constant area and a common centre of gravity, the triple focus of C.

‡ This remarkable curve has many beautiful properties and has been much studied, especially by the French. A bibliography and a proof of the above as well as other properties are given in my Dissertation, *loc. cit.* I have also given a bibliography in *L'Intermédiaire des Mathématiciens*, vol. 8 (1901), p. 10.

§ Cayley, *Educ. Times Reprint*, vol. 9 (1868).

‖ Wolstenholme, *Math. Problems*, *loc. cit.*, problems 1811, 1698, 1364, 1356.

Finally attention may be drawn to Problem 391 in an interesting collection by Ralph A. Roberts.*

10. It would be interesting, did space permit, to treat the nodal cubics (the reciprocals of tricuspidal quartics), whose equations may be thrown into the form†

$$X^{\frac{1}{3}} + Y^{\frac{1}{3}} + Z^{\frac{1}{3}} = 0,$$

by a method similar to that which we have used in the foregoing pages.

The results of the present paper were obtained four years ago. Within the past eighteen months a book by A. B. Basset‡ has appeared, making special note of the value of the method of projection in connection with the study of higher plane curves, whereas previously this theory had been largely confined to the treatment of conics.

SACKVILLE, N. B.
APRIL, 1903.

NOTE ON A PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER§

BY E. D. ROE, JR.

THIS note contains an elementary proof of the following familiar theorem :||

A necessary and sufficient condition that a function $f(x_1, x_2, \ldots, x_n)$ be a solution of the equation

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \cdots + \frac{\partial f}{\partial x_n} = 0, \tag{1}$$

is that f be a function of the differences of the x's.

---

* *A Collection of Examples and Problems on Conics and some of the Higher Plane Curves*, Dublin, 1882.

† Salmon, *loc. cit.*, p. 184.

‡ *An Elementary Treatise on Cubic and Quartic Curves*, Cambridge, 1901.

§ Presented to the American Mathematical Society at the Evanston meeting, September 2–3, 1902

|| *Cf.* Faà di Bruno's *Binäre Formen*, p 29 and p. 46, where, however, only the sufficient condition, not the necessary condition, is proved. *Cf.* also G. Salmon, *Higher Algebra*, 4th edit., §§63–64; and Gordan and Noether, in *Math. Annalen*, Vol. 10 (1876), pp. 549–552.

Proof. Consider any function $f(x_1, x_2, \ldots, x_n)$. Introducing the new variables $y_1, y_2, \ldots, y_n$, by the relations

$$\begin{array}{ll} x_1 &= y_1 + y_2 + \cdots + y_{n-1} + y_n, \\ x_2 &= \phantom{y_1 + {}} y_2 + \cdots + y_{n-1} + y_n, \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_{n-1} &= \phantom{y_1 + y_2 + \cdots + {}} y_{n-1} + y_n, \\ x_n &= \phantom{y_1 + y_2 + \cdots + y_{n-1} + {}} y_n, \end{array} \tag{2}$$

we have

$$f(x_1, x_2, \cdots, x_n) = F(y_1, y_2, \cdots, y_{n-1}, y_n);$$

whence

$$\frac{\partial F}{\partial y_n} = \frac{\partial f}{\partial y_n} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_n} + \frac{\partial f}{\partial x_2}\frac{\partial x}{\partial y_n} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_n},$$

or, by (2),

$$\frac{\partial F}{\partial y_n} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \cdots + \frac{\partial f}{\partial x_n}. \tag{3}$$

Therefore f will satisfy the equation (1) when and only when $dF/dy_n = 0$; that is, when and only when F is a function of $y_1, y_2, \cdots, y_{n-1}$ alone. But from (2) we have at once,

$$y_1 = x_1 - x_2, \qquad y_2 = x_2 - x_3, \cdots, \qquad y_{n-1} = x_{n-1} - x_n;$$

these differences are independent, a relation

$$x_i - x_{i+1} = F(x_1 - x_2, \cdots, x_{i-1} - x_i, \quad x_{i+1} - x_{i+2}, \cdots, x_{n-1} - x_n)$$

between them being impossible, since to give all the variables $x_1, x_2, \cdots, x_i$ the same increment, while $x_{i+1}, \cdots, x_n$ receive none, would lead to contradiction. Any other differences of the x's, as $x_i - x_j\,(j \neq i+1)$, can however be expressed in terms of the former differences by the relation

$$x_i - x_j = y_i + y_{i+1} + \cdots + y_{j-1}.$$

Hence the theorem is true.*

---

* The theorem in question is a special case of the following:

If $\phi_1, \phi_2, \cdots, \phi_{n-1}$ are $n-1$ independent solutions of the equation

$$X_1\frac{\partial f}{\partial x_1} + X_2\frac{\partial f}{\partial x_2} + \cdots + X_n\frac{\partial f}{\partial x_n} = 0,$$

(where the coefficients are any functions of the x's), then the necessary and sufficient condition that a function f be a solution of the equation is that f be a function of the ϕ's. See, for example, E Goursat, *Équations aux dérivées partielles du premier ordre* (1891), pp. 31–32.

It will be seen that this proof does not use the theory of determinants, or the theory of the Jacobian, nor is any knowledge of the properties of differential equations required.

If in equation (1) we replace x_i by x_i/a_i $(i = 1, 2, \cdots, n)$, we have the more general theorem that

A function $f(x_1, x_2, \cdots, x_n)$ *will be a solution of the equation*

$$a_1\frac{\partial f}{\partial x_1} + a_2\frac{\partial f}{\partial x_2} + \cdots + a_n\frac{\partial f}{\partial x_n} = 0, \tag{4}$$

when and only when f *is a function* of the $n-1$ independent differences $x_i/a_i - x_{i+1}/a_{i+1}$, or, more simply, *of the* $n-1$ *independent determinants* $a_{i+1}x_i - a_ix_{i+1}$.

For example, since the resultant, R, of two binary forms

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0,$$
$$b_0x^n + b_1x^{n-1} + \cdots + b_n = 0,$$

satisfies the partial differential equation

$$b_0\frac{\partial R}{\partial a_0} + b_1\frac{\partial R}{\partial a_1} + \cdots + b_n\frac{\partial R}{\partial a_n} = 0,$$

it follows that R is a function of the $n-1$ independent determinants $a_ib_{i+1} - a_{i+1}b_i$.

Syracuse University, Syracuse, N. Y.,
April, 1902.

ON A GENERALIZATION OF THE SET OF ASSOCIATED MINIMUM SURFACES*

BY ARTHUR SULLIVAN GALE

1. Introduction. The well known memoirs on minimum surfaces by Sophus Lie† are based on the theorem that a minimum surface is a surface of translation whose generators are its minimum lines. As the generators are therefore imaginary curves the problem is suggested of considering all surfaces of translation whose generators are imaginary; the set of surfaces considered in this note consists of such surfaces.

By a *surface of translation* is meant a surface whose equations have the form

$$S: \quad \begin{aligned} x &= A(u) + A_1(v), \\ y &= B(u) + B_1(v), \\ z &= C(u) + C_1(v), \end{aligned}$$

for such a surface may be generated by translating either of the curves

$$\Gamma: \quad \begin{aligned} x &= A(u), \\ y &= B(u), \\ z &= C(u), \end{aligned} \qquad \text{or} \qquad \Gamma_1: \quad \begin{aligned} x &= A_1(v), \\ y &= B_1(v), \\ z &= C_1(v), \end{aligned}$$

so that a point of that curve describes the other. Through each point of S there will pass two curves on S which are congruent and parallel‡ to Γ and Γ_1 respectively; the totality of these curves may be spoken of as the two sets of *generators* of the surface.

* Presented to the American Mathematical Society, February 28, 1903.

† *Math. Ann.*, vol. 14 (1879), p. 331 and vol. 15 (1880), p. 465. In these papers Lie has developed the elements of the theory of surfaces of translation, and derived certain properties of minimum surfaces, which Darboux (*Théorie des surfaces*, particularly §§ 220–225, 234, 239) has clarified and amplified in his elegant treatment of that subject. Inasmuch as these properties do not depend on the minimum property, but are enjoyed by all surfaces of translation, it is to be regretted that Darboux did not develop them in connection with the sections dealing with surfaces of translation as such (§§ 81, 82, 84). The surfaces of translation have received but passing reference in other text books; *e. g.*, Niewenglowski, *Géométrie analytique*, vol. 3, p. 159; Bianchi, *Differentialgeometrie* p. 112; Scheffers, *Anwendung der Differential- und Integralrechnung auf Geometrie*, vol. 2, p. 188.

‡ Two congruent curves in space may be said to be *parallel*, if they may be brought into coincidence by a translation.

For some purposes it is convenient to regard S as the locus of the middle point of a chord whose extremities lie on the curves

$$x = 2A(u),\quad y = 2B(u),\quad z = 2C(u), \qquad \text{and} \qquad x = 2A_1(v),\quad y = 2B_1(v),\quad z = 2C_1(v).$$

In particular, this point of view shows how it is possible to generate a real surface by translating one imaginary curve along another; for the middle point of a line joining two conjugate imaginary points is real.

In what follows we shall confine ourselves to surfaces of translation whose generators are imaginary. Now the coordinates of an imaginary curve may be given in either one of two ways: 1°) by *complex functions* of a *real parameter*; 2°) by functions of a *complex parameter*.

The surfaces of translation generated by imaginary curves of the first kind are of little interest. For, if we have two curves

$$x = f_1(u) + if_2(u),\quad y = g_1(u) + ig_2(u),\quad z = h_1(u) + ih_2(u), \qquad \text{and} \qquad x = f_3(v) + if_4(v),\quad y = g_3(v) + ig_4(v),\quad z = h_3(v) + ih_4(v),$$

where the f_i, g_i and h_i are real functions of the real parameters u and v, the surface of translation generated by them can be real only when the conditions

$$f_2(u) + f_4(v) = 0,$$
$$g_2(u) + g_4(v) = 0,$$
and
$$h_2(u) + h_4(v) = 0,$$

are satisfied identically. Unless the six functions involved in these conditions reduce to constants which satisfy the conditions, the real points of the surface will consist at most of the points of a real curve; and if that reduction take place the surface may be generated by the real curves

$$x = f_1(u),\quad y = g_1(u),\quad z = h_1(u), \qquad \text{and} \qquad x = f_3(v),\quad y = g_3(v),\quad z = h_3(v).$$

We shall, therefore, consider only those surfaces whose generators are given by functions of complex parameters u and v, and we suppose that these functions are analytic.

In discussing surfaces whose generators are imaginary, the first question is naturally concerning their reality. A sufficient condition that S be real when Γ and Γ_1 are imaginary is evidently that $A_1(v)$, $B_1(v)$, and $C_1(v)$ be the conjugate imaginary functions* of $A(u)$, $B(u)$, and $C(u)$ respectively. And if we set $u = \alpha + i\beta$ and $v = \alpha - i\beta$, we see that this condition is also necessary; for if S be real, the imaginary parts of $A_1(v)$, $B_1(v)$, and $C_1(v)$ must be the imaginary parts, with their signs changed, of $A(u)$, $B(u)$, and $C(u)$ respectively, and hence the real parts of these functions can differ only by additive constants.† Hence:

A surface of translation whose generators are imaginary curves given in terms of conjugate imaginary parameters is real when and only when the generators of one set are the conjugate imaginary curves of the generators of the other set.

From the standpoint of differential equations, the surfaces of translation are evidently those surfaces whose coordinates satisfy the equation

$$\frac{\partial^2\theta}{\partial u \partial v} = 0.$$

If we set $u = \alpha + i\beta$ and $v = \alpha - i\beta$, this equation becomes

$$\frac{\partial^2\theta}{\partial \alpha^2} + \frac{\partial^2\theta}{\partial \beta^2} = 0.$$

Hence: *The analytic surfaces of translation whose generators are imaginary are identical with the surfaces whose coordinates satisfy Laplace's equation.*

Hence we may legitimately speak of such surfaces as *harmonic surfaces.* To write the equations of S in such a form that its coordinates satisfy Laplace's equation, we have only to set $u = \alpha + i\beta$ and $v = \alpha - i\beta$, when x,

* Two functions, $A(u)$ and $A_1(v)$, are said to be conjugate imaginary functions when they take on conjugate imaginary values for conjugate imaginary values of their arguments. $A(u)$ and $A_1(v)$ are the *same* functions of their respective arguments only when $A(u)$ is real for real values of u; but if $A(u) = a_1(u) + ia_2(u)$ then $A_1(v) = a_1(v) - ia_2(v)$.

† *Cf.* Picard, *Traité d'analyse*, vol. 2, chap. 1, §6. The system S (in Picard's notation) is the condition that the function $u(x, y) + iv(x, y)$ of the argument $x + iy$ have a unique derivative; it is also, evidently, the condition that the function $u(x, y) - iv(x, y)$ of the argument $x - iy$ have a unique derivative, *i. e.*, be an analytic function of $x - iy$.

y, and z are given as the real parts of $2A(u)$, $2B(u)$, and $2C(u)$ respectively. Conversely, to write the equations of the surface

$$x = f_1(\alpha, \beta),$$
$$y = f_2(\alpha, \beta),$$
$$z = f_3(\alpha, \beta),$$

where
$$\frac{\partial^2 f_i}{\partial \alpha^2} + \frac{\partial^2 f_i}{\partial \beta^2} = 0, \qquad (i = 1, 2, 3),$$

as the equations of a surface of translation, we determine the three functions $g_i(\alpha, \beta)$, $(i = 1, 2, 3)$, so that

$$\tfrac{1}{2}[f_1(\alpha, \beta) + ig_1(\alpha, \beta)] = A(u),$$
$$\tfrac{1}{2}[f_2(\alpha, \beta) + ig_2(\alpha, \beta)] = B(u),$$

and
$$\tfrac{1}{2}[f_3(\alpha, \beta) + ig_3(\alpha, \beta)] = C(u),$$

where $u = \alpha + i\beta$: then the surface may be generated by translating the curve

$$x = A(u),$$
$$y = B(u),$$
$$z = C(u),$$

so that one of its points describes the conjugate imaginary curve.

The fundamental quantities of a surface of translation are:

$$E = \Sigma\left(\frac{\partial x}{\partial u}\right)^2 = \Sigma\Big(A'(u)\Big)^2,$$

$$F = \Sigma\,\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} = \Sigma\,A'(u)A_1'(v),$$

$$G = \Sigma\left(\frac{\partial x}{\partial v}\right)^2 = \Sigma\Big(A_1'(v)\Big)^2.$$

$$D = \frac{1}{\delta}\begin{vmatrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 y}{\partial u^2} & \frac{\partial^2 z}{\partial u^2} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \frac{1}{\delta}\begin{vmatrix} A''(u) & B''(u) & C''(u) \\ A'(u) & B'(u) & C'(u) \\ A_1'(v) & B_1'(v) & C_1'(v) \end{vmatrix}.$$

$$D' = \frac{1}{\delta}\begin{vmatrix} \frac{\partial^2 x}{\partial u \partial v} & \frac{\partial^2 y}{\partial u \partial v} & \frac{\partial^2 z}{\partial u \partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = 0,$$

$$D'' = \frac{1}{\delta}\begin{vmatrix} \frac{\partial^2 x}{\partial v^2} & \frac{\partial^2 y}{\partial v^2} & \frac{\partial^2 z}{\partial v^2} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \frac{1}{\delta}\begin{vmatrix} A_1''(v) & B_1''(v) & C_1''(v) \\ A'(u) & B'(u) & C'(u) \\ A_1'(v) & B_1'(v) & C_1'(v) \end{vmatrix},$$

where $\delta^2 = EG - F^2$.

Since $D' = 0$, the total curvature reduces to

$$k = \frac{1}{r_1 r_2} = \frac{DD''}{\delta^2};$$

and when $A(u)$, $B(u)$, and $C(u)$ are the conjugate functions of $A_1(v)$, $B_1(v)$, and $C_1(v)$ respectively, F will be real, G will be the conjugate of E, and D'' will be the negative conjugate of D (as is seen by interchanging the second and third rows of the determinant defining D''). Hence:

*The total curvature of any harmonic surface is everywhere negative.**

Remark. An investigation of the elementary questions presenting themselves in connection with the fundamental quantities of a surface of translation leads, with the exception of well known theorems on minimum surfaces and the theorem stated below, to negative results only; *e. g.*, the only surface of translation whose generators are its asymptotic lines is the plane.

If the generators form an orthogonal system they are the lines of curvature of the surface.

For $F = 0$ is the condition for orthogonality and we always have $D' = 0$. An illustration is afforded by any cylinder, if the generators be the right lines on the cylinder and their orthogonal trajectories; and, further, this property

* This theorem has been stated by Professor Maclay; see note at the end of the present article.

characterizes the cylinders. For the tangents to the generators of one set along the points of a generator of the other set are all parallel, and hence that generator can be orthogonal to the generators of the first set only when it is a plane curve whose plane is normal to the tangents to the generators of the first set. This being true for any two generators of the second set, whose planes, from the definition of S, must be parallel, the generators of the first set must be straight lines; unless, indeed, the generators of the second set be themselves straight lines. Hence:

The totality of cylinders may be defined as those surfaces of translation whose generators are their lines of curvature.

2. Associated Harmonic Surfaces. In seeking all minimum surfaces applicable upon a given real minimum surface

$$S:\quad \begin{aligned} x &= A(u) + A_1(v), \\ y &= B(u) + B_1(v), \\ z &= C(u) + C_1(v), \end{aligned} \qquad (E = G = 0)$$

one is led to the singly infinite set of *associated** surfaces

$$S_a:\quad \begin{aligned} x' &= e^{ia}A(u) + e^{-ia}A_1(v), \\ y' &= e^{ia}B(u) + e^{-ia}B_1(v), \\ z' &= e^{ia}C(u) + e^{-ia}C_1(v), \end{aligned}$$

where a is real: of these surfaces the adjoint surface of Bonnet, obtained by setting $a = \frac{\pi}{2}$,—viz:

$$S_0:\quad \begin{aligned} x_0 &= i[A(u) - A_1(v)], \\ y_0 &= i[B(u) - B_1(v)], \\ z_0 &= i[C(u) - C_1(v)],\text{—} \end{aligned}$$

is very intimately connected with S. In this section we shall consider the properties of S_a on the supposition that S is any harmonic surface;† the sur-

---

* Darboux, *l. c.*, book III, chap. V.

† If the generators of S be *real* the set of surfaces

$$\begin{aligned} x &= e^{a}A(u) + e^{-a}A_1(v), \\ y &= e^{a}B(u) + e^{-a}B_1(v), \\ z &= e^{a}C(u) + e^{-a}C_1(v), \end{aligned}$$

will possess the same properties as the set under discussion.

faces S_a may still be termed the surfaces associated with S, and S_0 the adjoint surface.

If E_a, F_a etc., denote the fundamental quantities of S_a, we have, in terms of those of S:

$$\begin{aligned} E_a &= e^{2ia}E, & D_a &= e^{ia}D, \\ F_a &= F, & D'_a &= D' = 0, \\ G_a &= e^{-2ia}G, & D''_a &= e^{-ia}D''. \end{aligned}$$

The following theorems are proved at once by means of these relations.

Corresponding areas of associated harmonic surfaces are equal.

For if $d\sigma$ and $d\sigma_a$ be the elements of surface of S and S_a respectively, we have

$$d\sigma_a = (E_a G_a - F_a^2)du\,dv = (EG - F^2)du\,dv = d\sigma.$$

The angles between the generators at corresponding points of associated harmonic surfaces are equal.

For if ω_a denote the angle between two generators of S_a we have

$$\cos\omega_a = \frac{F_a}{\sqrt{E_a G_a}} = \frac{F}{\sqrt{EG}} = \cos\omega.$$

Associated harmonic surfaces are applicable upon each other when and only when they are minimum surfaces.

For the square of the element of arc on S_a,—

$$ds_a^2 = e^{2ia}E du^2 + 2F du\,dv + e^{-2ia}G dv^2,\text{—}$$

is independent of a when and only when $E = G = 0$.

The total curvature of associated harmonic surfaces is the same at corresponding points.

For the total curvature is given by

$$k_a = \frac{D_a D''_a}{\delta_a^2} = \frac{DD''}{\delta^2} = k.$$

Associated harmonic surfaces have the same spherical representation.

For the fundamental quantities of the spherical representation,—viz:

$$e_\alpha = -d_\alpha = \frac{G_\alpha D_\alpha^2}{\delta_\alpha^2} = \frac{GD^2}{\delta^2},$$

$$f_\alpha = -d'_\alpha = -\frac{F_\alpha D_\alpha D''_\alpha}{\delta_\alpha^2} = -\frac{FDD''}{\delta^2},$$

$$g_\alpha = -d''_\alpha = \frac{E_\alpha D''^2_\alpha}{\delta_\alpha} = \frac{ED''^2}{\delta^2}, \text{—}$$

are independent of α.

The coordinates of S_α may be easily expressed in terms of the coordinates of S and those of the adjoint surface S_0. For

$$e^{\pm i\alpha} = \cos\alpha \pm i \sin\alpha,$$

and hence[*]

$$x' = x\cos\alpha + x_0 \sin\alpha,$$

$$y' = y\cos\alpha + y_0 \sin\alpha,$$

and

$$z' = z\cos\alpha + z_0 \sin\alpha,$$

In view of this fact we see that, although the mean curvature of S_α cannot be expressed simply in terms of that of S, yet it may be expressed simply in terms of the mean curvatures of S and S_0, as follows:

$$h_\alpha = \frac{e^{i\alpha}ED'' + e^{-i\alpha}GD}{\delta^2} = h\cos\alpha + h_0 \sin\alpha.$$

NOTE. In a paper presented to the American Mathematical Society on Feb. 22, 1902, Professor Maclay considered the following singly infinite set of harmonic surfaces:

$$x' = x + x_0 \tan\alpha,$$

$$(1) \qquad y' = y + y_0 \tan\alpha,$$

$$z' = z + z_0 \tan\alpha,$$

where x, y, z, x_0, y_0, and z_0 are the coordinates of corresponding points of any harmonic surface and the adjoint surface. These surfaces are similar by

[*] *Cf.* Darboux, *l. c.*, § 211.

pairs to those treated above, the ratio of similitude being $\cos\alpha$. Although many of the theorems given above are analogous to those given by Professor Maclay, I had been led to consider those surfaces before hearing his paper. If the surface with which Professor Maclay starts be written as a surface of translation:

$$x = A(u) + A_1(v),$$
$$y = B(u) + B_1(v),$$
$$z = C(u) + C_1(v),$$

then the family he considered may be written:

$$(2)\qquad \begin{aligned} x' &= (1 + i\tan\alpha)A(u) + (1 - i\tan\alpha)A_1(v),\\ y' &= (1 + i\tan\alpha)B(u) + (1 - i\tan\alpha)B_1(v),\\ z' &= (1 + i\tan\alpha)C(u) + (1 - i\tan\alpha)C_1(v). \end{aligned}$$

It seems to me that this set of surfaces may be more readily studied in the form (2) than in the form (1): for in the form (1) the fundamental quantities of any surface of the set are found in terms of those of two surfaces, while in the form (2) they may be expressed in terms of those of a single surface.

YALE UNIVERSITY,
AUGUST, 1902.

TWISTED QUARTIC CURVES OF THE FIRST SPECIES AND CERTAIN COVARIANT QUARTICS*

By H. S. White

The advantages of representing points of a plane cubic curve by values of an elliptic integral of the first kind are generally understood. That method gives the simplest exposition of the configurations associated with the inflexional points; of the theory of residuation; of the Steinerian Polygons; and of the linear construction of the cubic from pairs of conjugate points, which we owe to Schroeter. The plane cubic is the first curve (properly so called) in the indefinitely extended series of Elliptic Normal Curves—so named by Klein. The next in order is the twisted quartic in three dimensions, the intersection-line of two quadric surfaces. The application of elliptic parameters to such quartics is perhaps less familiar, though even more advantageous than in the lower case. What, for example, could be more elegant than Harnack's linear construction of conjugate triplets of points upon the curve from two given triplets of the same kind? Or where can better geometric illustrations be found for the problems of multiplication and division of elliptic arguments than in the relations of systems of quadric surfaces in the sheaf passing through the curve?

I propose here to use this parametric representation of points on the curve as an aid to proving the existence of certain irrational covariant curves and rational covariant surfaces.

1. The Index of a Quadric Surface, and of a Derived Quartic. The points in which a plane touches the curve and cuts it again are denoted, according to Abel's Theorem, by the parameters

$$-a+b, \quad a, \quad -a-b,$$

since

$$(-a+b)+(2a)+(-a-b)=0.$$

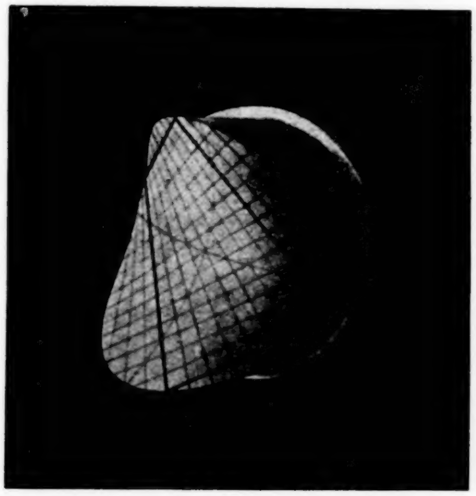
A chain of points such that a tangent plane in each can cut the curve in

* Read before the Chicago Section of the American Mathematical Society, Dec. 30, 1897.

the point which precedes and the point which follows may therefore be denoted by the parameters:

$$\begin{array}{ccccccccccc} \ldots & & a-2b & & a & & a+2b & & a+4b & & \ldots \\ \ldots & -a+3b & & -a+b & & -a-b & & -a-3b & & \ldots & \end{array}$$

Lines joining two consecutive points, with the positive coefficient of b the greater, will cut every line joining consecutive points of the opposite sort; for the sum of the four arguments will be zero. The two sorts of join-lines are therefore generators in the two reguli of a quadric surface containing the curve. Upon the quadric we shall consider a gauche polygon inscribed in the



curve. Depending on the nature of the quadric which supports it, this polygon will prove to be either indefinite or closed. Since generators of the same sheaf meet nowhere, a polygon can close only with some even number of sides, $2n$. If it closes with $2n$ sides, let us say that n is the *index* of the supporting quadric with respect to the curve. Among the sheaf of supporting quadrics, those of finite index will be rare, those of no index or of infinite index, the rule.

It is of course well known that the index is the same, whatever point of the curve be chosen as the first point of the polygon, and whichever of the

two generators meeting there be taken as the first side; for the condition of closing is

$$a + 2nb \equiv a \qquad \text{(modulis the periods } \omega_1, \omega_2\text{)},$$

or

$$2nb = m_1\omega_1 + m_2\omega_2 = P,$$

whence

$$b = \frac{P}{2n}:$$

that is, the index is n if the argument b is a primitive $2n$th part of a period.

The Vossian quadrics are the six whose index is 2; which support therefore inscribed quadrilaterals. A model, at least approximately correct, of one such surface is Brill's No. 15, 3rd Series, showing a hyperbolic paraboloid on which the quartic curve is cut out by a cylinder. (See figure.) For each kind of quartic curve of the first species the number of real Vossian quadrics is discussed by Harnack, in volume 12 of the *Mathematische Annalen*.

A new system of curves is discovered if, while regarding a particular quadric of the sheaf, whose index we may think of as infinite, we suppose the sides of every inscribed gauche polygon to be produced indefinitely. Starting from any point, the sides may be called the first odd side, first even side, second odd, second even, etc. The intersection of a first odd side with its second even side has for its locus another curve upon the same surface;—an algebraic curve, for the determining conditions are algebraic, and of the fourth order, since each generator cuts it twice. Call this the C_2 of the given surface, and by analogy call the fundamental quartic the C_1. Now in the same way define C_3 as the locus of the intersection of each first odd side with its third even side, and so may be defined an endless series of quartic curves: C_4, C_5, ..., upon the surface of infinite index.

Suppose however that the quadric has finite index, n. Then evidently the C_2 and the C_{n-2} will be the same quartic, also C_3 and C_{n-3}, etc. Hence the index of any curve of the system may be taken less than $(n+1)/2$, where n is the index of the supporting quadric. There is a distinction between quadrics of even index and those of odd index; for the latter contain among the concomitant quartics always one of index $(n+1)/2$ which degenerates evidently into a doubly-counting plane section, having however as many tangent generators as the C_1. As regards the number of derivative quartic curves on any surface, and the derivatives of derivatives, the analogy of regular polygons inscribed in a circle and the related star polygons is sufficiently close to allow trustworthy use of geometric imagination.

One thing it concerns us to note: that the definitions of auxiliary figures involve only projectively invariant relations—tangency, coplanarity, intersections; and that beyond those the relations of curve to surface, and of curve to curve in the surface, are denoted by pure numbers; *that therefore all relations here discussed are invariant with respect to collineation in three-dimensional space.*

2. Variable C_2, C_3, C_n as Characteristics of new Covariant Surfaces. On every quadric containing the fundamental quartic curve there exist quartics corresponding to all integral indices. On some the series is finite, and the numbers recur in regular order as their index describes the series of natural numbers; the six Vossian Quadrics showing only the fundamental quartic itself as corresponding to every index. We may safely assume that continuous variation of the supporting quadric causes each derivative quartic C_2, C_3, . . . C_n to describe a continuous locus, an algebraic surface. It is possible to calculate the degree and predict the singular points of each such surface in advance of the labor of finding its equation. One example will show the method; we shall discuss the locus of the first derivative quartic C_2.

Two surfaces of the sheaf have in common no points save those of the fundamental quartic. Hence the locus of the C_2 can have no conical points or other singular points outside the C_1. Each secant of the C_1 intersects its own C_2 in two points, and meets no other C_2. Hence the order of the locus is 2, increased by the number of intersections to be counted upon the C_1. Now the only surfaces on which the C_1 is at the same time a C_2 are the Vossian surfaces, six in number. *Hence the fundamental quartic must be a sextuple line upon the desired locus, and in it the locus must be tangent to each of the Vossian surfaces.* The order is then: $2 + 6 \cdot 2 = 14$. There are 12 quadrics having index 3 with respect to the curve. On each of these the C_2 is a plane conic twice counted. Twelve quadric surfaces of the sheaf therefore are tangent to the 14-ic locus along as many conics.

The next surface, the locus of curves C_3, is of order 38, and has the C_1 for an 18-fold line, along which it is tangent to the 6 Vossians and the 12 quadrics of period 3 respectively. The numerical characteristics increase rapidly. To extend the list could offer no difficulty. From the considerations 1) that each locus of a C_n is uniquely determinate and 2) that it is covariantly connected with the fundamental curve and not dependent upon any one quadric of the sheaf in distinction from any other, we draw the conclusion:

There exist simultaneous covariants which are combinants of the sheaf of quadrics containing the fundamental quartic, possessing the orders and singularities computed by the foregoing method.

3. On the Calculation of Equations of Derivative Quartics. The fundamental C_1 is projected from four points by cones. From one of these points the generators of any one of the quadrics are projected as tangents to a second cone. The inscribed polygon is projected into any plane as a polygon inscribed in one conic and circumscribed about a second. Salmon has given formulae for the locus of intersection of any first side with its third, fourth, fifth, etc.; and they appear as conics of a sheaf, whose equations are rational covariants of the equations of the two conics. To reckon the equations for a C_2, it would be best to assume the C_1 given by two quadric equations in canonical form:

$$F = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = 0,$$
$$\Phi = A_1x_1^2 + A_2x_2^2 + A_3x_3^2 + A_4x_4^2 = 0.$$

For any quadric of the sheaf: $F + \lambda\Phi = 0$, the projection of its contour from one of the four cone-vertices is readily written out, together with the equation of the conic into which the quartic is projected. Then by Salmon's rules the trace of the C_2 can be found, and between its equation and $F + \lambda\Phi = 0$ the parameter λ may be eliminated. The eliminant will contain an unsymmetrical factor, which may be thrown out by the usual process, leaving the equation of the surface whose characteristic is a C_2. Similarly any other C_n and its locus could be represented algebraically, and, if desired, in terms of fundamental combinants in case of the surfaces.

The most inviting problem connected with these associated quartic curves is perhaps this: upon a quadric of given index, to find what relation subsists either between the elliptic moduli of the C_1 and the C_n, or between the invariant anharmonic ratios belonging to these two quartic curves.

Northwestern University,
Evanston, Ill.

ON THE CHARACTERISTICS OF DIFFERENTIAL EQUATIONS*

By E. R. Hedrick

Introduction

The importance of the theory of characteristics in the study of differential equations is well known to all who are interested in that subject. The ordinary developments† are, however, somewhat lacking in rigor. It is the purpose of this paper to present, in somewhat altered form, a new method for the introduction and treatment of characteristics, devised by Hilbert and given by him in lectures at Göttingen 1900–1901.

In developing the method, no previous knowledge of the theory will be assumed, but attention will be called to well known results when they appear.

Besides the new introduction thus proposed, a few theorems new to the theory will be given.‡

Part I. Partial Differential Equations of the First Order.

1. The Cauchy-Kowalewski Theorem. The process to be developed depends directly on a theorem due to Cauchy§ and Kowalewski,‖ which will be assumed as known in the following form:

* This paper includes two papers read before the American Mathematical Society: (1) "On the Characteristics of Partial Differential Equations," read Dec. 28, 1901; (2) "On the Integral Curves and Strips of Partial Differential Equations," read Feb. 28, 1903.

† See Goursat, *Éq. aux dér. part. 1 et 2 ordres;* Fricke, *Functionentheoretische Vorlesungen;* von Weber, *Encyc. d. Math. Wiss.* II. A, 5; Monge, *Applications de l'analyse à la géométrie*, 1795 (Edition Liouville 1850); etc.

‡ The theorems of §§ 2, 3, 5, 6 of Part I, and §§ 2, 3 of Part II are for the most part due to Hilbert; but some of the alterations made are essential. The work of §§ 2, 3, in each Part is somewhat altered; that of § 6, Part I is somewhat generalized, and is extended in § 5, Part II. § 4, Part I, and § 4, Part II, together with those parts of the previous sections which lead up to these sections, were not given by Hilbert, and are believed to be new.

§ *Oeuvres complètes de Cauchy*, 1re serie, vol. 7.

‖ *Crelle*, vol. 80.

Theorem. Let us consider any partial differential equation of the first order in one dependent and two independent variables:

$$(1) \qquad F(x, y, z, p, q) = 0,$$

where $p = \partial z/\partial x$, $q = \partial z/\partial y$; and let this equation be solved for p:

$$(2) \qquad p = f(x, y, z, q).$$

Then there exists one and only one analytic solution

$$z = \nu(x, y)$$

($i. e.$ one and only one analytic function of x and y which when substituted for z renders (2) an identity in x and y) *which satisfies the boundary condition*

$$(3) \qquad z\big]_{x=x_0} = \nu(x_0, y) = \phi(y),$$

where $\phi(y)$ is a preassigned analytic function of y near $y = y_0$; provided that $f(x, y, z, q)$ is a single valued analytic function of x, y, z, q ($i. e.$ developable in absolutely convergent power series by Taylor's theorem) *in the neighborhood of the point $x = x_0$, $y = y_0$, $z = z_0 = \nu(x_0, y_0) = \phi(y_0)$, $q = q_0 = \partial\nu(x_0, y_0)/\partial y_0 = \phi'(y_0)$, and that $\nu(x, y)$ is analytic* near $x = x_0$, $y = y_0$.*

The equation (1) can be put in the form (2), and the function $f(x, y, z, q)$ will have the desired properties mentioned above, if $F(x, y, z, p, q)$ is an analytic function of x, y, z, p, q near the point $x = x_0$, $y = y_0$, $z = z_0$, $p = p_0$, $q = q_0$ where $F(x_0, y_0, z_0, p_0, q_0) = 0$, and if further $\partial F/\partial p \neq 0$ at this point. In this case (2) is called the *normal form* of the differential equation (1). But the solution (2) may be many valued, and in such case the theorem will hold for each single valued solution separately. Corresponding to each single valued solution of the form (2) there will then exist one and only one solution $z = \nu(x, y)$ of (1) which passes through the curve

$$\begin{cases} z = \phi(y), \\ x = x_0. \end{cases}$$

* This requirement that the solution be analytic is not necessary: see E. R. Hedrick, *Dissertation*, Göttingen, 1901, p. 19.

in the interval $y_1 < y < y_2$, provided the function $F(x, y, z, p, q)$ is analytic and $\partial F/\partial p \neq 0$ in this interval, for the values of z, p, q given by $z = \phi(y)$, $q = \phi'(y)$, $q = \psi(y)$, where $F[x_0, y, \phi(y), \phi'(y), \psi(y)] = 0$; and this solution $v(x, y)$ will be analytic in the interval for sufficiently small values of $|x - x_0|$.*

Geometrically the theorem then states that one and only one analytic integral surface of (1), corresponding to each single valued solution of the form (2), can be found, upon which lies a given analytic curve in a plane parallel to the YOZ plane, provided F is analytic and $\partial F/\partial p \neq 0$ for the values of x, y, z along the curve and the corresponding values of p and q given by (1) and by the equation of the tangent to the curve.

Similar statements of course hold on interchange throughout of x and y, and simultaneous interchange of p and q.

By a suitable transformation it is clear that, *in general*, any curve in space would uniquely determine a corresponding integral surface. This would not follow from the above theorem, however, in case the transformation which carries the given curve into the curve (3) transforms the equation (1) into an equation which cannot be thrown into the normal form (2). It is precisely this consideration, interesting for its own sake, which will eventually lead us to the characteristics themselves, and to the so-called integral curves. We shall find that the curves (which we shall call *integral curves*) for which the Cauchy-Kowalewski process may fail, in the above sense, are divided into two sorts, in general: (1) those which lie on integral surfaces, which we shall call *characteristic curves*; (2) those which do not lie on any integral surface. We shall then see that in general several integral surfaces pass through a characteristic curve. Hence the Cauchy-Kowalewski theorem is actually untrue for any of the integral curves, including both the sorts mentioned above, for in neither case does the curve determine an integral surface through itself, uniquely.

We proceed at once to the determination of the characteristic curves. We shall find first the characteristic curves on a given integral surface, and later all the characteristic curves in space, *i. e.*, all those curves which lie on integral surfaces and for which the Cauchy-Kowalewski process may fail, in the above sense.

After having established some of the principal properties of characteristic

* See preceding footnote.

curves, we shall determine the integral curves in general; and then state a few general theorems based upon the foregoing results.

2. Failure of the Cauchy-Kowalewski Process. Let us then first assume given an integral surface

$$z = \nu(x, y), \tag{5}$$

where $\nu(x, y)$ is a known analytic function of x and y which satisfies the equation (1). By the above reasoning we see that any curve on this surface will, in general, determine an integral surface uniquely. We wish to find first the characteristic curves on this surface, *i. e.* the curves for which this conclusion cannot be drawn from the Cauchy-Kowalewski theorem, in the above sense. Aside from the curves already considered in §1, the analytic curves on the surface (5) may be represented by the equations

$$\begin{cases} y = \lambda(x), \\ z = \nu(x, y), \end{cases} \quad \text{or} \quad \begin{cases} y = \lambda(x), \\ z = \nu[x, \lambda(x)] = \mu(x), \end{cases} \tag{6}$$

where $\lambda(x)$, and hence also $\mu(x)$, are analytic functions of x.

Let us now transform the equation (1) by the transformation

$$\begin{cases} x_1 = y - \lambda(x), \\ y_1 = x, \end{cases} \quad \text{or} \quad \begin{cases} x = y_1, \\ y = x_1 + \lambda(y_1), \end{cases} \tag{7}$$

in order to transform our boundary curve (6) into the form

$$\begin{cases} x_1 = 0, \\ z = \mu(y_1), \end{cases} \tag{8}$$

which is more suitable for the application of the Cauchy-Kowalewski theorem. In the new variables x_1, y_1 the function z will have derivatives $\partial z/\partial x_1$, $\partial z/\partial y_1$, which we shall call p_1, q_1 respectively. From (7) we have at once:

$$\begin{cases} p = -p_1\lambda'(x) + q_1, \\ q = p_1, \end{cases} \quad \text{or} \quad \begin{cases} p_1 = q, \\ q_1 = p + q\lambda'(y_1). \end{cases} \tag{9}$$

where $\lambda'(x) = d\lambda(x)/dx$. Hence the given differential equation (1) becomes

$$(10) \qquad F[y_1, x_1 + \lambda(y_1), z, q_1 - p_1 \lambda'(x), p_1] = 0.$$

The Cauchy-Kowalewski theorem will then be applicable to the curve (8), provided F is analytic and $\partial F/\partial p_1 \neq 0$ for each set of values of x_1, y_1, z_1, p_1, q_1 in a certain interval given by $x_1 = 0$, $y_1 = y_1$, $z = \mu(y_1)$, $p_1 = \psi(y_1)$, $q_1 = \mu'(y_1) = d\mu(y_1)/dy_1$, where

$$F[y_1, 0 + \lambda(y_1), \mu(y_1), \mu'(y_1) - \psi(y_1)\lambda'(y_1), \psi(y_1)] = 0.$$

The same result is evidently obtained by writing $\partial F/\partial p_1$ as a function of x, y, z, p, q and substituting $x (= y_1) = x$, $y (= x_1 + \lambda y_1) = 0 + \lambda(x)$, $z = \mu(x)$, $p[= q_1 - p_1 \lambda'(x)] = \phi_1(x)$, $q(= p_1) = \phi_2(x)$, where

$$F[x, \lambda(x), \mu(x), \phi_1(x), \phi_2(x)] = 0$$

and where $\phi_1(x) = \mu'(x) - \phi_2(x)\lambda'(x)$. But again this same result is obtained by substituting $z = \nu(x, y)$, $p = \nu_x(x, y)$, $q = \nu_y(x, y)$ — where $\nu_x(x, y) = \partial\nu(x, y)/\partial x$ and $\nu_y(x, y) = \partial\nu(x, y)/\partial y$ — and then $y = \lambda(x)$. For $\nu[x, \lambda(x)] = \mu(x)$ and the quantities substituted for p and q satisfy the relations for $\phi_1(x)$ and $\phi_2(x)$, since

$$\mu'(x) = \nu_x[x, \lambda(x)] + \nu_y[x, \lambda(x)] \cdot \lambda'(x)$$

and $F[x, y, \nu(x, y), \nu_x, y_y] = 0$ identically in x and y, because (5) is a solution of (1).

The curves on the surface (5) for which the Cauchy-Kowalewski process may fail, in the above sense, are then given by the equation:

$$(11) \qquad \frac{\partial F}{\partial p_1} \equiv -\frac{\partial F}{\partial p}\frac{d\lambda(x)}{dx} + \frac{\partial F}{\partial q} = 0,$$

where z, p, q are to be replaced by $\nu(x, y)$, $\nu_x(x, y)$, $\nu_y(x, y)$, respectively. For, by the above, this equation becomes an identity in x, for any curve on (5) for which the Cauchy-Kowalewski process might fail in the above sense, when these substitutions are made and then $\lambda(x)$ is substituted for y. But $\nu(x, y)$ is a known analytic function. Hence, after the substitution of $\nu(x, y)$

throughout for z, the equation (11) becomes an ordinary differential equation for $\lambda(x)$, *i. e.* for y as a function of x:

$$(11a)\qquad -F_p(x,y)\frac{dy}{dx}+F_q(x,y)=0,$$

where F_p and F_q denote the functions of x and y which result from the substitution of $\nu(x,y)$ throughout for z in $\partial F/\partial p$ and $\partial F/\partial q$ respectively. The equation (11a) has one and only one solution corresponding to any preassigned set of values of x and y, in a region where the coefficients are analytic and $F_p \neq 0$; and this solution is an analytic function of x. Hence:

THEOREM. *There exists on the given surface* (5) *a one parameter family of curves of the type* (6), *satisfying* (11a), *one and only one of which passes through any given point of the surface, in a region where F_p and F_q are analytic and $F_p \neq 0$.*

For these curves the Cauchy-Kowalewski process fails, in the above sense; *i. e.*, we cannot assert that one of these curves determines uniquely the integral surface on which it lies. *That it actually does not, remains to be proved.* As in §2, we shall call these curves *characteristics* or *characteristic curves*, since by definition they lie on an integral surface. The equation (11a) will be recognized as the principal equation of the characteristic curves, as ordinarily derived.* It follows that *every integral surface is completely covered by characteristic curves, one and only one of which passes through any given point of the surface, provided F_p and F_q are analytic and $F_p \neq 0$ at that point.* And we see that unless F is not analytic near some point given by the substitution of $x=x$, $y=\lambda(x)$, $z=\nu[x,\lambda(x)]$, $p=\nu_x[x,\lambda(x)]$, $q=\nu_y[x,\lambda(x)]$ found above, the Cauchy-Kowalewski theorem directly applies. Excluding such exceptional points (where F is not analytic) we have the

THEOREM: *Any curve of the type* (6) *which lies on an integral surface and is not a characteristic curve, determines uniquely the integral surface on which it lies, under the above restriction.*

We have tacitly assumed that F_p does not vanish identically in x and y along the surface (5). If it does, we can easily modify our work so that x and y are interchanged unless F_q also vanishes identically. If now F_p and F_q both vanish identically along the given surface (5), the equation (11) is

* See Goursat, *l. c.*, *I ordre*, p. 115; Fricke, *l. c.*, p. 491; and others.

identically satisfied for any curve whatever on the surface, and the Cauchy-Kowalewski process may fail for every such curve.

We are thus led to enquire into the nature of the surfaces obtained by eliminating p and q between the three equations

$$(12) \qquad F = 0, \quad F_p = 0, \quad F_q = 0.$$

If, as a result of such elimination a surface $z = \nu(x, y)$ is found, which satisfies (1) and for which F_p and F_q both vanish identically in x and y, then such an integral surface must be excluded in the statement of the theorems above, and in fact ***every curve on such a surface would satisfy the definition of a characteristic curve.*** It is an advantage of the present method that our attention is called at once to such surfaces, for it turns out that they are precisely the so-called ***singular solutions.**** We shall exclude them from our discussion for the present, however, in view of their evident peculiarity, in order that the statements of theorems may remain simple.

Likewise, if F_p vanishes at any point (x_0, y_0) of the surface (5), the theorems of this section hold true unless F_q vanishes at the same point. But if both F_p and F_q vanish at any point (x_0, y_0, z_0) of the surface (5), then we cannot assert that one and only one solution of (11a) passes through this point, and such points must be excluded from the discussion in the statement of the theorems. They will in general be points upon the envelope of the one parameter family of solutions of (11a), or in particular the point into which such an envelope may degenerate. Such points, or their locus, may be found by substituting $\nu(x, y)$ throughout for z in $\partial F/\partial p$ and $\partial F/\partial q$ and solving the resulting equations $F_p(x, y) = 0$, $F_q(x, y) = 0$ as simultaneous equations for x and y. This will give, in general, discrete points in the (x, y) plane, which are singular points for the surface (5), in the above sense.

It now remains to determine the characteristics without assuming $\nu(x, y)$ as known, and this we shall do in §3. On the other hand, it is clear that we have used the assumption that the curve (6) lies on the integral surface (5) only to show that the substitution of $\nu(x, y)$ for z and of $\nu_x[x, \lambda(x)]$ and $\nu_y[x, \lambda(x)]$ for p and q respectively was equivalent to the substitutions for z_1, p, q_1 which were to be made in F_{p_1}. We cannot make the above substitu-

* See Darboux, "Mémoire sur les solutions singulières," *Jour. sav. étr.*, vol. 27 (1880): Goursat, *l. c.*, *1 ordre*, p. 86; Forsyth, *Differential Equations*, p. 296; etc.

tion, of course, when we do not assume that the curve for which the process is to fail, lies on some integral surface. We shall then return to this point in §4, and seek to determine the *integral curves* in general, *i. e.*, whether they lie on an integral surface or not. The characteristic curves are evidently also integral curves.

3. The Characteristic Curves in Space. We have defined a characteristic curve to be a curve which lies on some integral surface and for which the Cauchy-Kowalewski theorem may fail in the above sense. We shall now proceed to prove that the theorem actually does not hold, *i. e.*, that there are other integral surfaces through this same curve.

We shall first obtain the differential equations of the characteristic curves, which by definition lie on an integral surface, without assuming the integral surface to be given. Since $\nu(x, y)$ is a solution of (1), that equation will become an identity in x and y when $\nu(x, y)$ is substituted for z. Likewise the two functions dF/dx and dF/dy will vanish identically when $\nu(x, y)$ is substituted for z:

$$(13)\qquad \begin{cases} \dfrac{dF}{dx} \equiv F_p p_x + F_q q_x + F_z p + F_x = 0, \\[2mm] \dfrac{dF}{dy} \equiv F_p p_y + F_q q_y + F_z q + F_y = 0, \end{cases}$$

where $p_y = dp/dy = \partial^2 z/\partial x \partial y = \partial q/\partial x = q_x$, and so on.

We have also the identities

$$(14)\qquad \begin{cases} \dfrac{dz}{dx} = p + q\dfrac{dy}{dx}, \\[2mm] \dfrac{dp}{dx} = p_x + p_y\dfrac{dy}{dx}, \\[2mm] \dfrac{dq}{dx} = q_x + q_y\dfrac{dy}{dx}, \end{cases}$$

where $z = \nu(x, y)$, and $y = \lambda(x)$, any function.

The equations (1), (11), (13), (14) are satisfied identically in x, when $\nu(x, y)$ is substituted for z and then $\lambda(x)$ for y, where $\nu(x, y)$ is a solution of (1), and $\lambda(x)$ is a corresponding solution of (11); or, what is the same

thing, these equations (1), (11), (13), (14) are satisfied identically in x by the equations $z = \mu(x)$, $y = \lambda(x)$, the equations of the desired characteristic curves. Hence (1), (11), (13), (14) are the differential equations of the characteristic curves. From them we can eliminate p_x, $p_y(= q_x)$, and q_y, so as to obtain the equations:

$$(15)\qquad \begin{cases} \dfrac{dy}{dx} = \dfrac{F_q}{F_p}, & \dfrac{dp}{dx} = -\dfrac{F_x + pF}{F_p}, \\[2ex] \dfrac{dz}{dx} = \dfrac{pF_p + qF_q}{F_p}, & \dfrac{dq}{dx} = -\dfrac{F_y + qF_z}{F_p}, \end{cases}$$

which, together with (1), must be satisfied by $y = \lambda(x)$, $z = \nu[x, \lambda(x)] = \mu(x)$, $p = \left.\dfrac{\partial\nu(x, y)}{\partial x}\right]_{y = \lambda(x)} = \phi_1(x)$, and $q = \left.\dfrac{\partial\nu(x, y)}{\partial y}\right]_{y = \lambda(x)} = \phi_2(x)$, say.

The equations (15) are then a set of four simultaneous ordinary differential equations for the determination of y, z, p, q as functions of x. The equation (1) must, however, also be satisfied, and hence one of the equations (15) (say the last), may be regarded as supernumerary, and the value of q (say) may be found from (1), after y, z, p have been determined by the first three of equations (15).

If, now, we think of x, y, z, p, q all as functions of a new auxiliary variable t, we shall have $\dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = \dfrac{dy}{dt}$, etc., whence the equations (15) become:*

$$(16)\qquad \begin{cases} \dfrac{dx}{dt} = F_p, & \dfrac{dy}{dt} = F_q, \qquad \dfrac{dz}{dt} = pF_p + qF_q, \\[2ex] \dfrac{dp}{dt} = -(F_x + pF_z), & \dfrac{dq}{dt} = -(F_y + qF_z), \end{cases}$$

of which the first equation is merely a definition of t. The set (15) [or (16)] is seen at once to be the set of equations ordinarily obtained for characteristics.†

* This form is also obtained if we exchange x and y in all our work. Or we might obtain it directly by altering the work slightly. It is convenient in avoiding an interchange of x and y when F_p vanishes.

† See Goursat, *l. c.*, *I ordre*, p. 115; Fricke, *l. c.*, p. 491; etc.

By the Cauchy-Kowalewski theorem for ordinary equations we know that if five constants x_0, y_0, z_0, p_0, q_0 are given and if

$$x]_{t=0} = x_0;\quad y]_{t=0} = y_0;\quad z]_{t=0} = z_0;\quad p]_{t=0} = p_0;\quad q]_{t=0} = q_0$$

be assumed as initial values, then a set of solutions of the equations (16) is uniquely determined:*

$$(17)\qquad \begin{cases} x = f_1\,(t,\ x_0,\ y_0,\ z_0,\ p_0,\ q_0) \\ y = f_2\,(t,\ x_0,\ y_0,\ z_0,\ p_0,\ q_0) \\ z = f_3\,(t,\ x_0,\ y_0,\ z_0,\ p_0,\ q_0) \\ p = f_4\,(t,\ x_0,\ y_0,\ z_0,\ p_0,\ q_0) \\ q = f_5\,(t,\ x_0,\ y_0,\ z_0,\ p_0,\ q_0) \end{cases}$$

and the five functions so obtained are analytic functions of x_0, y_0, z_0, p_0, q_0 as well as of t; provided only that the right hand sides of the given equations [as is the case in equations (16)] are analytic functions of all the variables present, near the point x_0, y_0, z_0, p_0, q_0.

The first three of equations (17) define a curve in space. The last two define, at each point of this curve, a tangent plane to the curve, by (14). This configuration of a curve in space together with tangent planes affixed at every point, we shall call a *strip*. Such a strip is, by the above, uniquely determined as the solution of equations (16) [or (15)] when any point (x_0, y_0, z_0) and a plane (p_0, q_0) through that point are given. This configuration of a point together with a plane through it, is called a *surface element*.

Since for our purposes x, y, z, p and q must, when found as functions of t, render the equation (1) an identity in t, it will be necessary to assume that x_0, y_0, z_0, p_0, q_0 satisfy (1), since these are to be the values of x, y, z, p, and q corresponding to $t = 0$. We shall then have $F(x_0, y_0, z_0, p_0, q_0) = 0$, or the surface element $(x_0, y_0, z_0, p_0, q_0)$ lies on an integral surface. Let us then assume that x_0, y_0, z_0 are the coordinates of a point on some integral surface, and that p_0, q_0 are the directions of that integral surface, at (x_0, y_0, z_0). We have seen in §2 that there exists one and only one characteristic line, on the integral surface, through any point of the integral surface; and with it are associated the uniquely determined tangent planes which are also tangent to

* See Picard, *Traité d'analyse*, vol. 2, p. 304; Fricke, *l. c.*, p. 435; etc.

the given integral surface. Such a configuration of a characteristic line together with the appended tangent planes belonging to the integral surface we shall naturally call a *characteristic strip*. But we have just seen that through any surface element $(x_0, y_0, z_0, p_0, q_0)$ there goes one and only one strip satisfying the equations (16). But these equations merely express the condition for a characteristic line on some integral surface, together with the appended tangent planes uniquely determined at every point by this integral surface. For they are constructed out of equation (11) and the identities (13) and (14). From this follows directly the

THEOREM: *Any surface element* $(x_0, y_0, z_0, p_0, q_0)$ *lying on an integral surface uniquely determines a characteristic strip, which lies wholly on the surface.*

For, if it did not lie wholly on the surface, then the characteristic strip given by §2 would be a second characteristic strip through the given surface element and lying on the integral surface, the existence of which is impossible.

Or, if x_0, y_0, z_0, p_0, q_0 *be a set of values satisfying the given differential equation* (1), *then the set of solutions of* (16) [or (15)]: x, y, z, p, q, *given by* (17), *satisfy the equation* (1) *identically in* t.*

There is a three parameter family of characteristic strips determined by the equations (16) [or (15)]. For, of the five constants x_0, y_0, z_0, p_0, q_0 in (17), one is determined by the relation $F(x_0, y_0, z_0, p_0, q_0) = 0$, and we get the same characteristic strip for each surface element of the one parameter family of surface elements lying along it. This leaves a three parameter family of characteristic strips in all space.

This three parameter family of strips will in general lie along a three parameter family of characteristic curves, given by the first three of equations (17). But it may happen that several characteristic strips lie along the same characteristic curve, so that there may be less than three parameters entering into the equations of the characteristic curves. In particular, for an equation of the form

$$A(x, y, z)p + B(x, y, z)q = C(x, y, z)$$

the first two of equations (15) [or the first three of (16)] do not involve p or q, and can be solved separately. In this case, then, the first three of equations

* The analytic proof is also immediate. See, *e. g.*, Fricke, *l. c.*, p. 433.

(17) involve only two essential parameters, and hence there is only a two parameter family of characteristic curves. If we solve all of equations (15) [or (16)] we shall find as before a three parameter family of characteristic strips, which are then arranged along the two parameter family of curves in such a way that through each curve there passes, in general, a one parameter family of strips. In general, then, any plane tangent to one of the curves determines a characteristic strip through the curve.

In general there is a three parameter family of characteristic curves with one characteristic strip along each curve, but in particular some of these may unite. For an equation linear in p and q, there is thus only a two parameter family of curves with a one parameter family of characteristic strips along each curve.

It is easily seen that an infinity of integral surfaces pass through any given surface element which satisfies (1). For instance, all curves, not integral curves, tangent to the element, determine uniquely integral surfaces through themselves, by the Cauchy-Kowalewski theorem. By the above, however, the unique characteristic strip determined by the given surface element lies on any integral surface through that element. Hence we have the following

THEOREM: *The characteristic curves, defined as those curves on integral surfaces for which the Cauchy-Kowalewski process may fail, actually do not determine the integral surfaces on which they lie. There are in fact an infinity of integral surfaces through every characteristic curve, and these surfaces are all tangent along this curve, at least provided they are tangent at one point.* For, the characteristic strip determined by their one common surface element, lies on each integral surface. The condition stated in the last line is not necessary, except when, as for linear equations, more than one strip lies along each characteristic line.

Let us now consider the curve (3); for which the Cauchy-Kowalewski theorem was stated in §1:

$$(18) \qquad \begin{cases} z = \phi(y) \\ x = x_0 \end{cases}$$

where $\phi(y)$ is analytic. This curve uniquely determines an integral surface of (1) if F is analytic and $\partial F/\partial p \neq 0$ in the interval considered for $x = x_0$, $y = y$, $z = \phi(y)$, $p = \psi(y)$, $q = \phi'(y)$, where $F[x_0, y, \phi(y), \psi(y), \phi'(y)] = 0$.

The curve (18) is not of the form (6), but we see that the Cauchy-Kowalewski theorem can fail only when $F_p = 0$ for the above substitutions, *i. e.*, for the functions $\phi(y)$ determined by (11), after dy/dx is replaced by $(dx/dy)^{-1}$; for since x_0 is constant, $dx/dy = 0$ for this curve. Hence any curve of the form (18) as well as of the form (6), for which the Cauchy-Kowalewski theorem fails, must satisfy (11), if, as above, we exclude points of the curve for which F is not analytic. If $F_p \equiv 1$, *i. e.*, if the equation is written in the form (2), it is clear that the only exceptional curves of the class (18) are those for which F(*i. e.* in this case f) is not analytic, which is precisely a restatement of the Cauchy-Kowalewski theorem. If $F_p \equiv 0$ in x, y, z, p, q, that is, if p does not occur in F, it is clear that every curve of the type (18) would be such an exceptional curve.

If $F_p \equiv 0$, however, we can interchange x and y in all that follows, unless F_q also vanishes identically in x, y, z, p, q, in which case (1) ceases to be a differential equation.

Assuming then that $F_p \not\equiv 0$, excluding the curves (18) which satisfy (11), and restricting ourselves, for the present, to an interval where F is analytic along the curve, it follows directly from the Cauchy-Kowalewski theorem that the curve (18) determines uniquely an integral surface. At any point $x_0, y_0, z_0 = \psi(y_0)$ of this curve we shall have

$$q_0 = \frac{d\psi(y)}{dy}\Big]_{y=y_0}$$

while p_0 is determined from the relation $F(x_0, y_0, z_0, p_0, q_0) = 0$. Hence a surface element is determined at each point of the curve. This surface element, in turn, determines uniquely a characteristic strip through this point, and this characteristic strip lies on the integral surface determined by the given curve. If now we let the point move along the curve, the corresponding characteristic curve (with its strip) evidently sweeps out the integral surface. The analytic statement consists in setting

$$(19)\qquad \begin{cases} x_0 = x_0, \text{ constant}; & p_0 = \phi(\eta), \\ y_0 = \eta\ , \text{ variable}; & q_0 = \dfrac{d\psi(y)}{d(y)}\Big]_{y=\eta} = \psi'(\eta)\,. \\ z_0 = \psi(\eta), \text{ arbitrary}; & \end{cases}$$

where $\phi(\eta)$ is determined by the identity in η:

$$F[x_0, \eta, \psi(\eta), \phi(\eta), \psi'(\eta)] \equiv 0.$$

The single characteristic *curve* given by $y_0 = \eta$ for any constant value of η is, from (17),

$$(20) \qquad \begin{cases} x = f_1[t, x_0, \eta, \psi(\eta), p_0, \psi'(\eta)], \\ y = f_2[t, x_0, \eta, \psi(\eta), p_0, \psi'(\eta)], \\ z = f_3[t, x_0, \eta, \psi(\eta), p_0, \psi'(\eta)], \end{cases}$$

where p_0 is determined as a function of $\eta, \psi(\eta)$, and $\psi'(\eta)$, as above. The two similar equations derived from the last two of equations (17) define the directions of a tangent plane at each point of this curve. The equations (20), for *variable* η, for any function ψ, define a single *surface*, and this is an integral surface, by the above. For every choice of ψ we get one such surface, where t and η are parameters, p_0 having been obtained as above as a function of $\eta, \psi(\eta)$, and $\psi'(\eta)$: and these are all the regular integral surfaces of the given equation (1). Hence we obtain the following

THEOREM: *If we can obtain the characteristics, as the solution, in the form* (17), *of equations* (16) [*or* (15)], *we can at once write down the general solution of the given equation* (1) *in the form* (20), *involving in general one arbitrary function and its derivative.*

The importance of characteristics in the study of differential equations is well illustrated by the theorem just stated. But they do even more than aiding directly in the solution, by making clear the geometrical meaning of all parts of the subject. It is not the purpose of this paper to develop further the extensive theories connected with characteristic lines and strips, the ordinary theory being the same as ours would be from this point on.* Indeed we have passed over already with only short geometric proofs, two theorems (p. 134 and p. 132), of which rigorous analytical proofs can be given, those given by Fricke (*l. c.*, p. 433) being satisfactory.† It can be shown, in fact, that *the equations* (20) *represent a solution even when* $\phi(\eta)$ *is merely a continuous function which has a continuous first derivative*, though the proof will

* See, *e. g.*, Goursat, *l. c.*, *1 ordre*, Chap. IV.

† See also Goursat, *l. c.*, *1 ordre*, p. 110.

be essentially different from that sketched above, in that we cannot then infer *from the Cauchy-Kowalewski theorem* the *existence* of an integral surface through the given curve.*

It is now easy to show that *every analytic partial differential equation of the first order has non-analytic solutions.* For if $\psi(\eta)$ be taken non-analytic, it follows easily that the surface determined by (20) is non-analytic near this curve, in a two dimensional region. The details of the proof are given in the writer's thesis.†

4. The Integral Curves. Geometrical Interpretation. While our chief purpose has been attained in the discovery of the equations for the characteristics, it is interesting to discover the equations defining all those curves in space for which the Cauchy-Kowalewski theorem may not hold. These curves have been called (§1) the *integral curves.*

We shall exclude from the discussion, as above, on any curve,

$$(6a)\qquad \begin{cases} y = \lambda(x), \\ z = \mu(x), \end{cases}$$

any point $x_0 = x_0$, $y_0 = \lambda(x_0)$, $z_0 = \mu(x_0)$, if F is non-analytic about the point x_0, y_0, z_0, p_0, q_0, where $F(x_0, y_0, z_0, p_0, q_0) = 0$ and $\mu'(x_0) = p_0 + q_0\lambda'(x_0)$.

We shall then have to consider merely the possibility of the vanishing of $\partial F/\partial p$ for this curve, *i. e.* for the values $x = x$, $y = \lambda(x)$, $z = \mu(x)$, $p = \phi_1(x)$, $q = \phi_2(x)$, where $\phi_1(x)$ and $\phi_2(x)$ satisfy the equations $F[x, \lambda(x), \mu(x), \phi_1(x), \phi_2(x)] = 0$, and $\mu'(x) = \phi_1(x) + \phi_2(x)\lambda'(x)$. For, as remarked at the close of §2, these are exactly the substitutions we must make when we do not assume (as in §2) that the curve (6a) lies on an integral surface. If $\lambda(x)$ and $\mu(x)$ are unknown, p and q in (11) may then be replaced by the values found by solving the equations (1) and

$$(14a)\qquad \frac{d\mu(x)}{dx} = p + q\frac{d\lambda(x)}{dx}$$

* The proof given by Fricke, *l. c.*, p. 443, is immediately extensible to this case.

† See Hedrick, *l. c.*, pp. 20–25. This theorem is not extensible to systems of partial differential equations of the first order; see Hedrick, *l. c.*, p. 25.

as simultaneous equations for p and q. Any functions $\lambda(x)$ and $\mu(x)$ which satisfy the resulting equation

$$(11b)\qquad M\left(x, y, z, \frac{dz}{dx}, \frac{dy}{dx}\right) = 0,$$

when substituted for y and z respectively, will then evidently define an integral curve of the form $(6a)$. The equation $(11b)$ is due to Monge, and is called the *Monge equation*. In general it will be determined by eliminating p and q between the equations (1), (11), and $(14a)$, written in the form

$$\frac{dz}{dx} = p + q\frac{dy}{dx}.$$

The equation $(11b)$ is a single ordinary differential equation for the determination of z and y as functions of x. In general its solutions cannot be expressed by a finite number of parameters, for if any analytic function of x be substituted for z, for instance, there will still be in general a one parameter family of solutions for y, which, with the above arbitrary value of z, satisfy the equation. Among the numerous solutions will be included, of course, the characteristic curves, by definition. And it is otherwise evident that any solution of (15) will also satisfy $(11b)$.

The values of p and q corresponding to given values of x, y, and z, which we have found it necessary to substitute here (and in §2) are found by solving (1) and $(14a)$ as simultaneous equations. Now (1) constitutes for any fixed point (x, y, z) one relation between p and q, and defines at this point a one parameter family of planes (p, q) which in general envelop a cone whose vertex is in the point (x, y, z). It is clear that these are precisely the possible tangent planes to any integral surface through (x, y, z). Hence any integral surface must at every point be tangent to the cone through that point; and this condition is also sufficient. This *cone field* is then the exact geometrical counterpart of the differential equation. The conditions (11) and $(14a)$ mean geometrically that the integral curve is tangent at every point to the cone at that point.* We obtained the characteristics however,

* This geometrical analogy between integral surfaces and integral curves constitutes the justification of the latter terminology.

under these same conditions, with the additional requirement that the curve should lie on an integral surface. Hence we have the

THEOREM. *The integral curves are those curves in space which are tangent at every point to the cone field determined by the equation* (1) ; *and the characteristic curves are those integral curves which lie on integral surfaces.* It is this geometrical fact which is usually made the basis of the whole theory.

For the linear equation

$$A(x, y, z,)\, p + B(x, y, z,) q = C(x, y, z)$$

however, the general propositions are materially altered. For (11) becomes

$$A(x, y, z)\frac{dy}{dx} = B(x, y, z), \qquad (11c)$$

which does not involve p or q. It is therefore unnecessary to make the substitutions which were made in general, and any curve of the form (6a) which satisfies (11c) is an integral curve. If for y any function of x, say $\lambda(x)$, is substituted the equation (11c) will in general define a cylinder perpendicular to the plane of XOZ. *There is, therefore in general, one integral curve whose projection on the XOY plane is an arbitrarily assigned curve.* Or, if for z any function of x and y be substituted, the equation (11c) will reduce to an ordinary differential equation for y. *On any surface in space there is then a one parameter family of integral curves, which completely cover the surface. Of these, those which lie on integral surfaces are characteristic curves*; and these are, by §3, a two parameter family in space given by the first two of equations (14), of which the first coincides with (11c).

If, however A and B do not involve z, the facts are again changed. For then (11c) does not involve z, and hence becomes a single ordinary equation for y as a function of x. If $y = \lambda(x)$ satisfies this equation, the curve (6a) is an integral curve. *Hence, if A and B do not involve z, there is a one parameter family of curves in the XOY plane, such that any curve whose projection on this plane is a curve of this family is an integral curve. Of these curves, those which lie on integral surfaces are characteristic curves*; and these are given by solving the second of equations (15) :

$$A(x, y)\frac{dz}{dx} = C(x, y, z)$$

for z as a function of x, after replacing y by any solution $\lambda(x)$ of (11c).

It is easy to see that $(11c)$ gives the correct results not only for curves of the form $(6a)$ but also for any curve of the form (18).

It is evident in general that no integral surface will pass through an integral curve which is not a characteristic curve. And we have seen that a characteristic curve does not determine an integral surface uniquely. Hence:

THEOREM. *No integral curve determines uniquely an integral surface, there being more than one integral surface through any characteristic curve, and none at all through any other integral curve.*

Let us suppose the integral curves determined, in the form $(6a)$ [or (18)]. If we consider any other curve, C, of the form $(6a)$ [or (18)], it is clear that the Cauchy-Kowalewski theorem will hold, and that C will uniquely determine an analytic integral surface, for any portion of C for which F is analytic, and for which (11) does not hold at any point, when $\lambda(x)$, $\mu(x)$, $\phi_1(x)$, $\phi_2(x)$ are substituted for y, z, p, q, as above. But (11) can hold only at isolated points on C, since $\lambda(x)$ and $\mu(x)$ are supposed analytic, and C would coincide with an integral curve if (11) were satisfied at a set of points of C which cluster about any point of C. If we exclude these *isolated* points on C, for which (11) holds, any remaining portion of C determines uniquely an analytic integral surface through itself, provided F is analytic along this portion of C. The points excluded are, geometrically, the points at which C is tangent to an integral curve, since $\lambda'(x)$ and $\mu'(x)$ are used in the above substitution to determine $\phi_1(x)$ and $\phi_2(x)$. And it is clear that any constant value of x, y, z, $\lambda'(x)$, $\mu'(x)$ which satisfy $(11b)$ [or the corresponding equations from which $(11b)$ is derived] determine one or more solutions of $(11b)$; or of (11) under the above substitution.

If C lies on an analytic integral surface, it surely determines that integral surface uniquely; for each portion of it, between any two points which we have excluded, determines a portion of this same integral surface, uniquely. Hence, since the intervals about the excluded points may be made arbitrarily small, the whole curve C determines the whole surface uniquely. The only other cases which can arise are those in which C uniquely determines an integral surface, which is analytic along C except at the excluded points, and which is analytic in general except near an excluded point of C; or where the portions of C lying between excluded points determine pieces of *different* analytic integral surfaces. We may then state the

THEOREM: *Any curve C in space, of the form* $(6a)$ *or* (18), *which is not an integral curve, determines uniquely an integral surface which is analytic*

and which passes through the given curve, for any portion of C in which no point lies at which C is tangent to an integral curve, or at which $F(x, y, z, p, q)$ is non-analytic: the point x_0, y_0, z_0, p_0, q_0 of the curve C being given by $x = x$, $y = \lambda(x)$, $z = \mu(x)$, $p = \phi_1(x)$, $q = \phi_2(x)$, where $F[x, \lambda(x), \mu(x), \phi_1(x), \phi_2(x)] = 0$ and where $\mu'(x) = \phi_1(x) + \phi_2(x)\lambda'(x)$, for a curve of the form (6a); or by $x = x_0$, $y = y$, $z = \phi(y)$, $p = \psi(y)$, $q = \psi'(y)$, where $F[x_0, y, \phi(y), \psi(y), \phi'(y)] = 0$, for a curve of the form (18).

5. Possible Families of Characteristics. It is easily (and usually)* shown that if

$$z = \phi(x, y, a, b), \tag{24}$$

represents any two parameter family of solutions† of (1), then the general solution of (1) is given by

$$\begin{cases} z = \phi[x, y, a, w(a)], \\ 0 = \phi_a + \phi_w w'(a), \end{cases} \tag{25}$$

where $w(a)$ is an arbitrary function; and the characteristic curves are given by

$$\begin{cases} z = \phi(x, y, a, b), \\ 0 = \phi_a(x, y, a, b) + c\phi_b(x, y, a, b), \end{cases} \tag{26}$$

as we see from equations (21) and (22).

We have seen that there is in general a three parameter family of characteristics of any partial differential equation of the first order. Let us now inquire if any three parameter family of curves may be the family of characteristic curves of some such differential equation. Let the equations

$$\begin{cases} h(x, y, z, a, b, c) = 0, \\ g(x, y, z, a, b, c) = 0 \end{cases} \tag{27}$$

---

* See *e. g.*, Forsyth, *Differential equations*, p. 294; Goursat, *l. c.*, *1 ordre*. Chap. V; etc.

† The necessary and sufficient condition that a and b in (24) be independent parameters is that

$$\frac{\partial^2\phi}{\partial x\partial a}\frac{\partial\phi}{\partial b} - \frac{\partial^2\phi}{\partial x\partial b}\frac{\partial\phi}{\partial a} \text{ and } \frac{\partial^2\phi}{\partial y\partial a}\frac{\partial\phi}{\partial b} - \frac{\partial^2\phi}{\partial y\partial b}\cdot\frac{\partial\phi}{\partial a}$$

do not both vanish identically.

be the equations of the three parameter family considered. Solving for z and c, as is in general possible, we have

$$(28)\qquad \begin{cases} z = k(x, y, a, b), \\ c = l(x, y, a, b), \end{cases}$$

in which form we may suppose the family given. If this family is to coincide with (26) we must have

$$\frac{\partial k(x, y, a, b)}{\partial a} + l(x, y, a, b)\frac{\partial k(x, y, a, b)}{\partial b} = 0.$$

The general solution of this differential equation for k we know to be

$$(29)\qquad k = w(k_0, x, y),$$

where $k_0 = k_0(x, y, a, b)$ is any particular solution and w is an arbitrary function. *Hence the family* (27) *is by no means arbitrary. But if the projections of the curves* (27) *on the XY plane, i.e., the second of equations* (28), *be given, then there is a whole class of differential equations of the first order which have characteristics with these given projections; and these equations go over into one another by the transformation $z' = w(z, x, y)$; i. e., the differential equation, the solutions, and the characteristic curves are carried into each other by this transformation.*

Again, if we eliminate p and q between the equation (1) and the first two of equations (15), so as to get the Monge equations:

$$(30)\qquad M\left(\frac{dy}{dx}, \frac{dz}{dx}, x, y, z\right) = 0,$$

then we know that the characteristic curves of the equation (1) are among the solutions of this Monge equation, which has in addition infinitely many more solutions. Given now a three parameter family of solutions of (30):

$$\begin{cases} y = \phi_1(x, c_1, c_2, c_3), \\ z = \phi_2(x, c_1, c_2, c_3), \end{cases}$$

which are *not* characteristics of (1), it follows readily that the Monge equation and the differential equation (1) itself are each uniquely determined by this three parameter family of integral curves. It follows that they are not integral curves for any other differential equation than (1), and hence certainly not characteristic curves of any other equation than (1). But they are not characteristic curves for the equation (1), by hypothesis. Hence they are not characteristic curves for any differential equation of the first order. *Hence, any three parameter family of integral curves which are not characteristic curves of the corresponding partial differential equation of the first order, are not the characteristic curves of any such equation.* We also see that the same three parameter family of curves cannot be the family of characteristic curves of two distinct partial differential equations of the first order; for, regarded as integral curves, they determine uniquely the differential equation. An instructive geometrical interpretation of these results is immediate.

7. Connection with the Calculus of Variations. The characteristics of a differential equation may be defined as the extremals of a certain problem of the calculus of variations. For, let us consider the problem* of rendering the following integral an extremum (maximum or minimum):

$$I = \int_{t_0}^{t_1} \frac{dz}{dt}\, dt, \tag{31}$$

under the auxiliary conditions:

$$z' = \frac{dz}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt} \equiv px' + qy', \tag{32}$$

where z' denotes dz/dt, and so on; and

$$F(x, y, z, p, q) = 0. \tag{33}$$

This problem is substantially to find that strip—equation (32) requires that the tangent plane be tangent to the curve—which is composed of integral surface elements (equation (33)), the steepness of which is an extremum.

* This problem is not reducible to a problem in ordinary maxima and minima because the auxiliary conditions involve x' and y'.

The first necessary conditions which must be satisfied by a solution of the above problem are the same as those for the problem of rendering the following integral an extremum :*

$$(34) \qquad I' = \int_{t_0}^{t_1} \Big\{ z' + \lambda(z' - px' - qy') + \mu F(x, y, z, p, q) \Big\} dt,$$

under no auxiliary conditions; where λ and μ are unknown functions of t. These conditions, commonly known as Lagrange's, are :

$$(35) \qquad \begin{cases} d(-\lambda p)/dt = \mu F_x, & 0 = -\lambda x' + \mu F_p, \\ d(-\lambda q)/dt = \mu F_y, & 0 = -\lambda y' + \mu F_q, \\ d(\lambda)/dt = \mu F_z, & z - px' + qy' = 0, \\ \qquad F(x, y, z, p, q) = 0. \end{cases}$$

If now we eliminate λ, $d\lambda/dt$ and μ from these equations, we get at once, as necessary conditions for the given problem :

$$(36) \qquad \begin{cases} \dfrac{dy}{dx} = \dfrac{F_q}{F_p}, & \dfrac{dp}{dx} = -\dfrac{F_x + pF_z}{F_p}, \\ \dfrac{dz}{dx} = \dfrac{pF_p + qF_q}{F_p}, & \dfrac{dq}{dx} = -\dfrac{F_y + qF_z}{F_p}, \\ \qquad F(x, y, z, p, q) = 0. \end{cases}$$

But these are precisely the equations (5) of the characteristic strips of the differential equation (1). Regarded as necessary conditions for the given problem of the calculus of variations, their solutions are called the *extremals* of the given problem. We have then the

THEOREM : *The characteristic strips of any partial differential equation (1) of the first order, are the extremals of the problem of the calculus of variations stated at the head of this section. Or, the characteristic strips are those*

---

* The same results may be obtained by replacing z' in (31) directly by its value from (32); then replacing p [or q] by its value obtained by solving (33); and computing the necessary conditions for the resulting integral. For the statements here made see Kneser, *Lehrbuch der Variationsrechnung*, p. 20 and p. 117; Pascal, *Variationsrechnung* (German translation by Schepp), p. 34 and p. 44; Mayer, *Math. Ann.*, vol. 26 (1885), p. 74.

strips which satisfy the given differential equation, the steepness of which is an extremum.†

It seems that this connection between the calculus of variations and the theory of characteristics may be of value to one or to both of these theories. If, for instance, we seek the common characteristics of two given differential equations,

$$F(x, y, z, p, q) = 0, \tag{37}$$

$$G(x, y, z, p, q) = 0, \tag{38}$$

in order to find their common solutions, in case any such exist, the equations of the common characteristics are evidently given by the first necessary conditions of the problem of rendering I in (31) an extremum, under the auxiliary conditions (32), (33) [or (37)], and (38).

Let us suppose the equations (37) [or (33)] and (38) to be solved for p and q:

$$p = \phi(x, y, z,), \tag{39}$$

$$q = \psi(x, y, z,), \tag{40}$$

in which form we may suppose (37) and (38) given. Let us then substitute these values of p and q in the equation (32), and the resulting value of z' in turn in the given integral (31). This latter becomes

$$I = \int_{x_0}^{x_1} \left\{ \phi(x, y, z) + \psi(x, y, z) y' \right\} dx, \tag{41}$$

with no auxiliary conditions; where now z is to be regarded as an unknown function of x and y. The first necessary condition for the existence of an extremum is then

$$\frac{d}{dx}(\psi) = \frac{\partial}{\partial y}(\phi + \psi y'), \tag{42}$$

or

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial\psi}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial\psi}{\partial z}\frac{\partial z}{\partial y}\frac{dy}{dx} = \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} + \frac{\partial\psi}{\partial z}\frac{\partial z}{\partial y}\frac{dy}{dx}.$$

† This second, geometric, statement is not rigorous, since the sufficient conditions of the calculus of variations may not be satisfied.

which reduces to

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial z}\phi = \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}\psi \tag{43}$$

when we take (39) and (40) into account. But this is precisely the necessary and sufficient condition for the existence of common characteristics of the given equations. Such a result indicates that the matter of this paragraph may be useful in simplifying certain considerations in the theory of characteristics.

8. Conclusion. Extension to other Equations. It is evident that the whole work up to this point may be extended without difficulty to the case of a single partial differential equation in one dependent and several independent variables. Such an extension will indeed present nothing essentially new. But the extension to equations of the second (or higher) order (or to the practically identical case of a set of equations of the first order) is essentially different in important particulars, and will for that reason be treated in the second part of this paper. The reason for such a distinction is that the solution of any single partial differential equation of the first order may be reduced to the solution of a set of ordinary differential equations, as we have done in the case of equation (1). But the solution of a partial differential equation of the second (or higher) order, or a set of two (or more) partial differential equations of the first order in two (or more) independent variables, cannot, in general, be reduced to the solution of a set of ordinary differential equations. It is obvious that such a fact must seriously affect the extension of the theory of characteristics to equations of the second order, for it was by the aid of the characteristics that we succeeded in reducing the solution of the equation (1) to the solution of the set (15) of ordinary differential equations. There remains however a considerable analogy between the above work and that for an equation of the second order, and an attempt will be made to abbreviate the following work wherever possible.

CONTENTS

ANNALS OF MATHEMATICS

Published in October, January, April, and July, under the auspices of Harvard University, Cambridge, Mass., U. S. A.

Cambridge: Address *The Annals of Mathematics*, 2 University Hall, Cambridge, Mass., U. S. A. Subscription price, $2 a volume (four numbers) in advance. Single numbers, 75c. All drafts and money orders should be made payable to Harvard University.

London: Longmans, Green & Co., 39 Paternoster Row. Price, 2 shillings a number.

Leipzig: Otto Harrassowitz, Querstrasse 14. Price, 2 marks a number.

PRINTED BY THE SALEM PRESS CO., SALEM, MASS., U. S. A.